Quasi-static aeroelasticity and flying boats

Aerodynamics, Hydrodynamics, stability, control . . .

Las Palmas June 2018
Ph. Destuynder

The french team for 2017 America's cup...

Outline

Hypotheses and goals(idem)

Hypotheses

- The reduce frequency f_r is small -> quasi-static approach ;
- The boat is rigid (but the mast is flexible for the future) ;
- The aerodynamical forces on the sails are coupled : jib and main sail ;
- The rudder is assumed to be rigid but articulated ;
- The foils have several degrees of freedom (pitch+rake and heave...) ;
- A control system can be included in the foils (in the example).

Goals for the future...

- Ensure the dynamic stability of the flying ship ;
- Define a control system in order to *fight* external perturbations ;
- To learn about flying ships : tacking, overtaking, waves, wake... ;
- Suggest a real time strategy for sailing to the skipper.

The aerodynamic center and the force center

On a rigid body the aero-hydro-forces are represented by a torsor.

Let us set : $\mathcal{T}(P) = (F, M(p))$ with $M(P) = M(0) + F \wedge OP$.

Definition (The force center)

The force center is define as the point P_c solution of : $M(P_c) = 0$. It doesn't exist in general ; it is not unique in 3D (but it is in 2D).

Definition (aer-hyd-ro center)

There is a point (non unique) where the euclidian norm of $M(P_f)$ is minimal. We call it the aero-hydro-center. It is solution of Details :

$$\min_{P \in \mathbb{R}^3} (F \wedge OP).(F \wedge OP) - 2(F \wedge M(0)).OP; \text{ and for uniqueness : } (F.OP_f)_3 = 0$$

Remark

Sometimes $P_c = P_f$ (good case always true in 2D for a boat, not for an aircraft because of the weight Details*)*

Basic property of P_f

Let assume that the forces applied are depending linearly on the parameters (for instance, an angle of attack α). This is a general situation.

Case of a single parameter

If one has : $F = \alpha F_0 \text{ and } M(0) = \alpha M_0$. Then P_f is unchanged with respect to α (simplify by α^2). At this point the moment is independent on α.

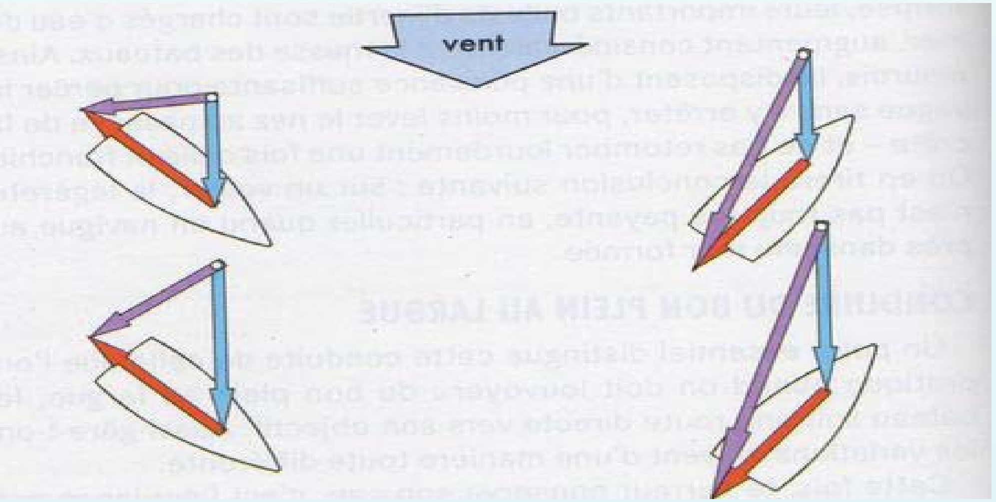
Case of several parameters (pitching α, rolling β and yawing γ)

Because the matrix and the right handside of the model characterizing P_f is a second order dependent with respect to this parameter the point P_f doesn't move at the first order in the vicinity of the origin.

Remark (Conclusion concerning P_f)

P_f is a nice point for the description of the movement of a rigid structure.

The apparent flow velocity at P_f



$$\boxed{V_a = V_{absolu} - V(P_f)}$$

Let (e_x, e_y, e_z) be an Eiffel frame for a given substructure. The three angles are :

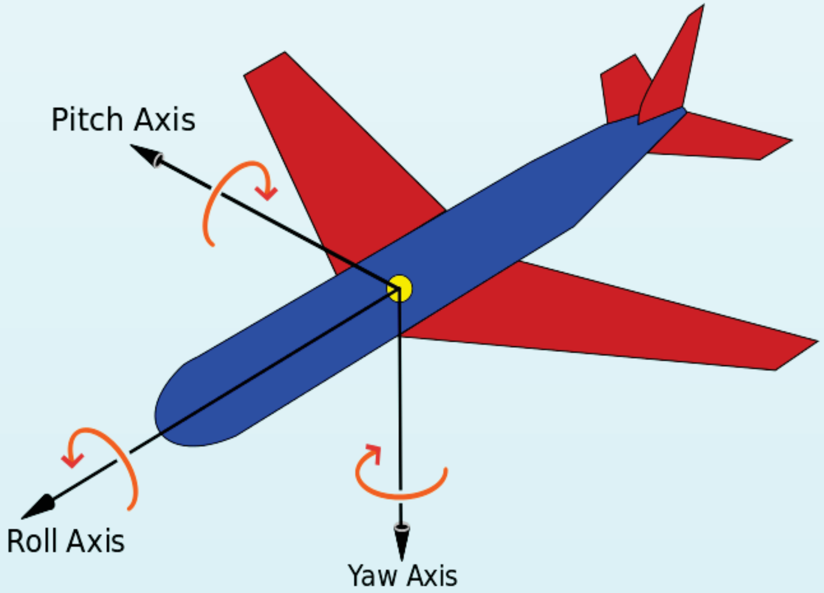
Pitch : $\alpha = -\arccos(\dfrac{(e_x . V_a)}{||V_a||_2})$,

Roll : $\beta = -\arccos(\dfrac{(e_z . V_a)}{||V_a||_2})$,

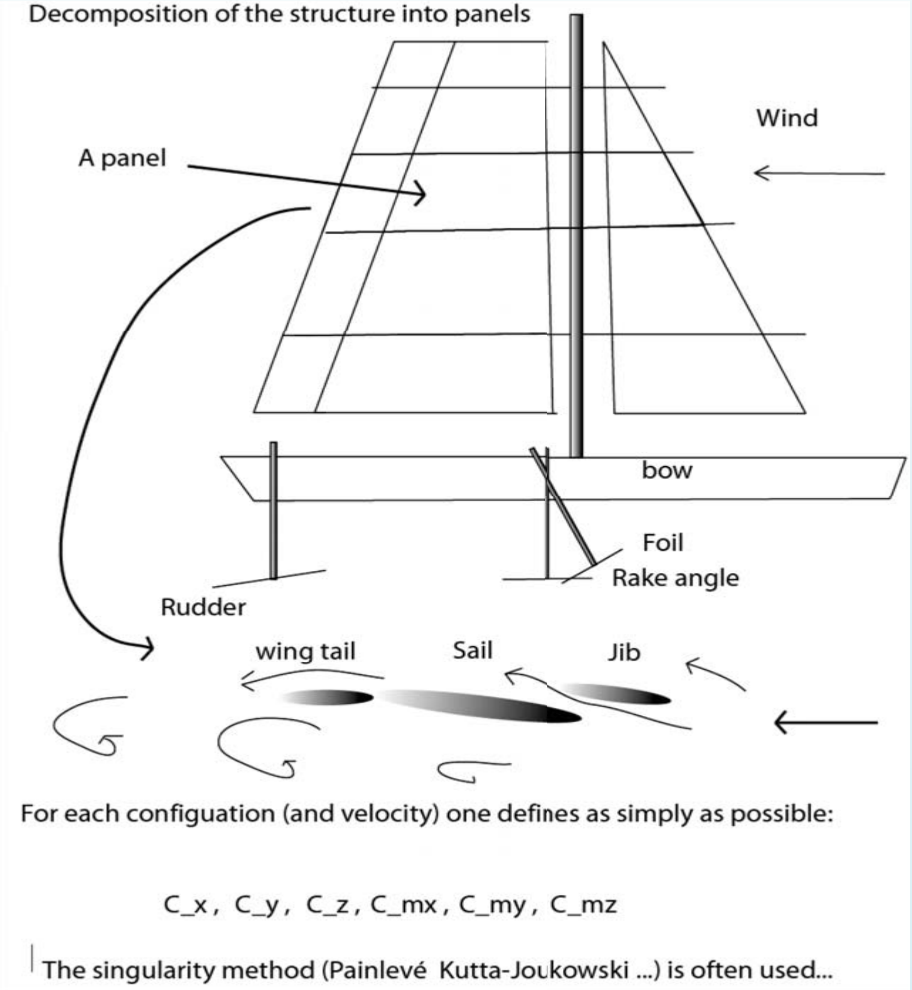
Yaw : $\gamma = -\arccos(\dfrac{(e_y . V_a)}{||V_a||_2})$.

$$\boxed{V_{absolu} = Ve_x}$$

$$||V_a||_2 = V\sqrt{1 + 2V(e_x . V(P_f)) + \frac{||V(P_f)||_2^2}{V^2}}$$



The panel method in 2D (poor but cheap and fast...)



The aerodynamical coefficients depend on the three angles α, β, γ

The model used in the panel method

$$\boxed{\begin{array}{l}\dfrac{\partial^2 \varphi}{\partial x^2} + \dfrac{\partial^2 \varphi}{\partial z^2} = 0 \text{ in } \Omega, \\[2ex] \dfrac{\partial \varphi}{\partial \nu} = 0 \text{ on } \Gamma_\infty, \quad \dfrac{\partial \varphi}{\partial \nu} = -V(\nu . e_x) \quad \text{on } \Gamma_S, \\[2ex] \int_{\Gamma_S} \varphi = 0 \text{ for instance.}\end{array}}$$

Then Expl (+ compressibility/viscosity ...)

$$\varphi = \sum_{i=1,P} K_{S_i} \sqrt{r} \cos(\frac{\theta}{2}) \eta(x, z) + \varphi_R.$$

$$\boxed{C_x(\Gamma_{si}) = -\frac{1}{SV^2} \int_{\Gamma_{si}} [|\frac{\partial \varphi}{\partial s}|^2](\nu . e_x) ds = \frac{\pi K_{S_i}^2}{2SV^2}(e_{S_i} . e_x)}$$

e_{S_i} unit vector of the trailing edge of Γ_{si}.

The Tacoma bridge

The story happened on the November 11 1940. This phenomenon is at the origin of the theory of the :

Aeroelasticity.

The story of the break down (Nasa)

There was a science professor (miracle !)

The notations

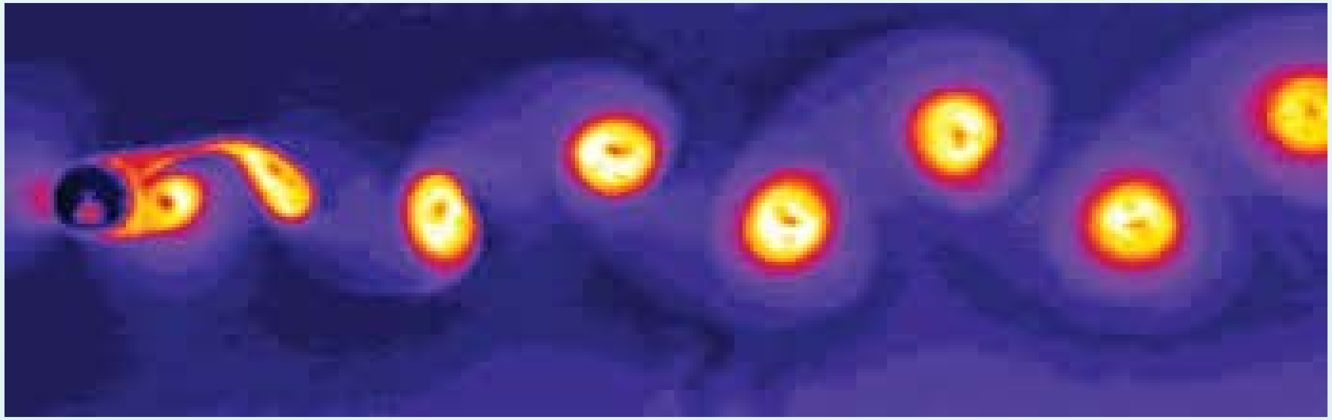
- Wind speed : $V \simeq 72\ km/h \simeq 20\ m/s$;
- Width of a cross section : $L \simeq 10\ m$;
- Strouhal number : $S_t \simeq .1$;
- Reynolds number ; $Re \simeq \frac{20 \times 10 \times 1.2}{2.6\ 10^{-5}} \simeq 10^7$;
- Reduce frequency : $f_r \simeq .05 << 1!$.

There was many explanations... but the right one came only in the late 70^{th}

1. Von Karman suggested that the accident was induced by famous vortices (but the Strouhal number is not correct according to the structure and the frequency, furthermore the Reynolds number is much too large for Strouhal instabilities !
2. The resonance mechanism which destroyed the bridge of Angers in 1860.
3. The classical flutter which occurs at a particular velocity when two (at least) eigenmodes have the same frequency ($\simeq$ 1980) but eliminated from a computational and experimental Japanese study.
4. The galloping (heaving movement discussed by J. Den Hartog was suggested in $\simeq$ 1980). It is certainly at the origin of the movement of the bridge.
5. The stall flutter in torsion (R. Scanlan $\simeq$ 1981) .

Von Karman (the VK-paths)

Let us consider a wire with diameter D. For low speed flow, ($R_e < 10^4$), one can observe boundary layer instabilities which are well organized.
These instabilities induce transverse forces at a Strouhal frequency ($\simeq .2\frac{U}{D}$), and for a H cross section as Tacoma : ($\simeq .1\frac{U}{D}$) or else 2 Hz.



Calcul NS-incomp.

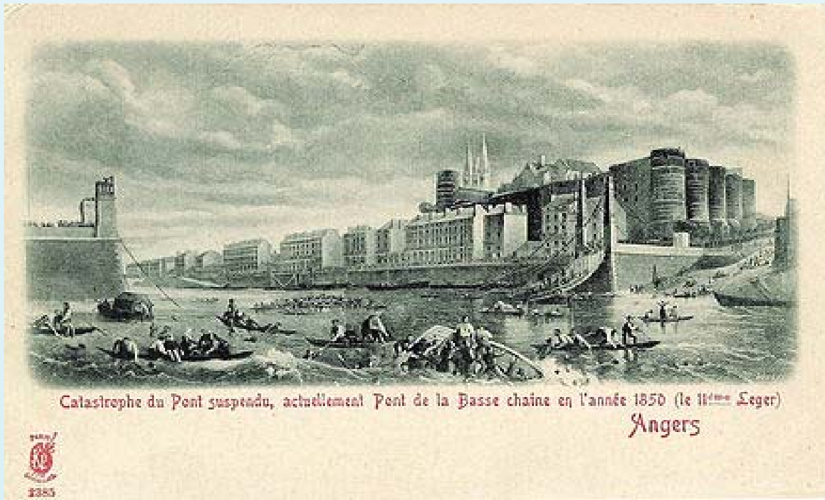
But the frequency observed on the movie of Tacoma collapse is $\simeq .2$ Hz.

Resonance

When the excitation frequency is close to an eigenfrequency of a flexible structure, one can observe a resonance phenomenon. The singing glass is an example.
In 1860 the bridge of Angers "discovered" the trick and collapse down.

"You have "to break the step" if you don't want to break the bridge"

Before

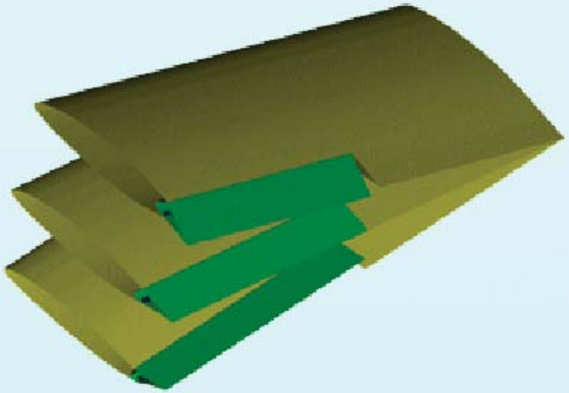


After

Classical flutter

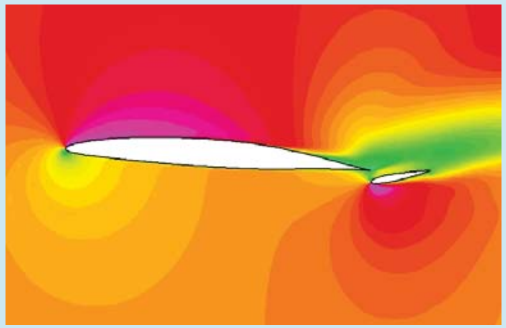
This phenomenon was discovered on american fighters *Wild Cat*, but was well understood few years latter. At the beginning engineers were tempted to make flutter responsible of any aerodynamical crashes.

The numerical simulation on a complex structure is not an easy job because of the violence of the instability. Nevertheless the NAL laboratory (japan) did a convincing study proving that the classical flutter was not responsible of Tacoma collapse....



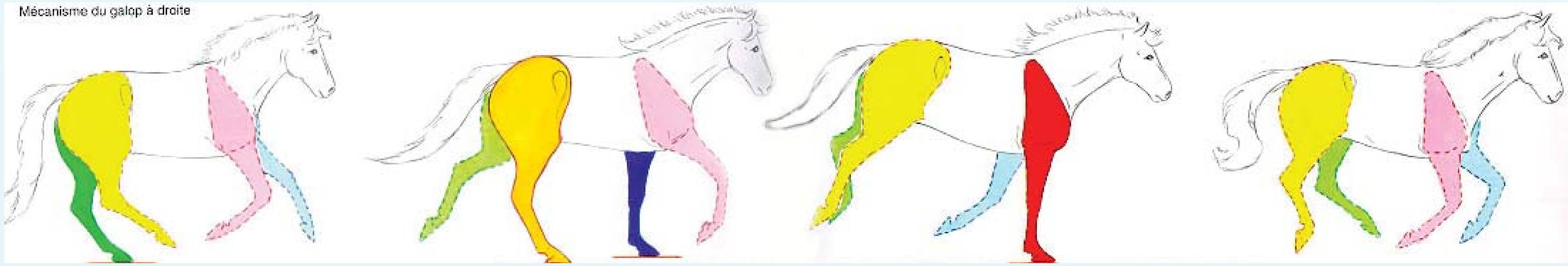
Flutter in situ

Flutter

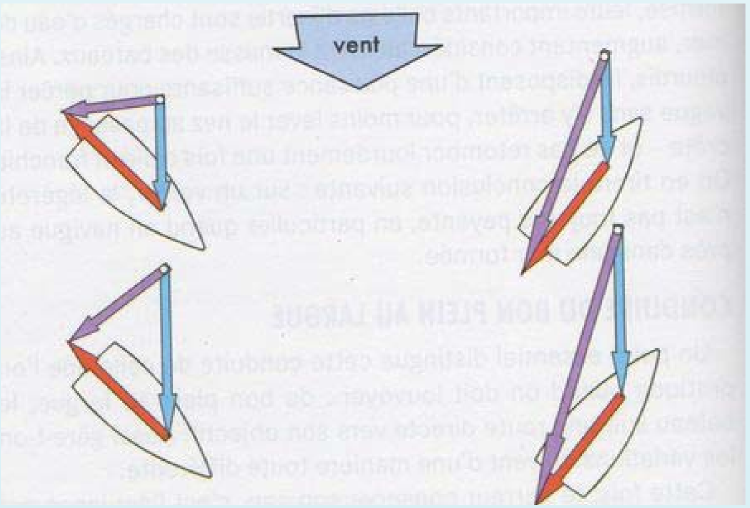


The galloping of a bridge

It is a vertical movement as the one of a horse back !



It creates an apparent wind : $V_a = V - u$,
V is the absolute wind and u the one of the structure.



J. Den Hartog criterion

The movement is a translation in the vertical direction : Jacob Den Hartog

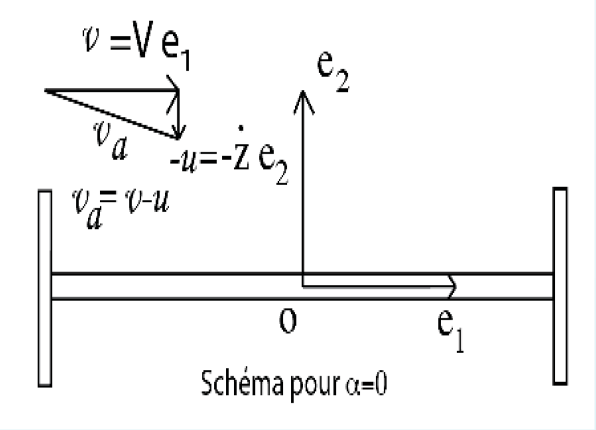
The model (α is the angle of attack at rest) :

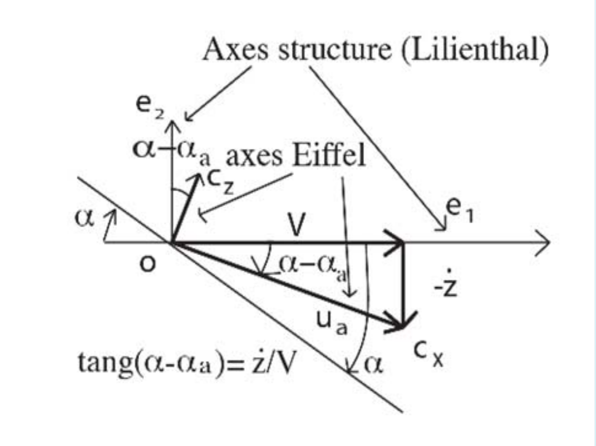
$$M\ddot{z} + Kz = f(z, \dot{z}), \; z(0) = \dot{z}(0) = 0,$$

and by linearizing for $(z, \dot{z})$ small :

$$M\ddot{z} - \frac{\partial f}{\partial \dot{z}}(0,0)\dot{z} + (K - \frac{\partial f}{\partial z}(0,0))z = f(0,0)$$

$$\begin{cases} f(z,\dot{z}) = \frac{\varrho S ||V_a||^2}{2} \\ [c_x(\alpha_a)\sin(\alpha_a - \alpha) + c_z(\alpha_a)\cos(\alpha_a - \alpha)] \\ \text{et} \quad , \; V_a = Ve_1 - \dot{z}e_2, \;\; ||V_a||^2 = V^2 + \dot{z}^2, \\ \alpha_a = \alpha - \text{arctang}(\frac{\dot{z}}{V}) \end{cases}$$





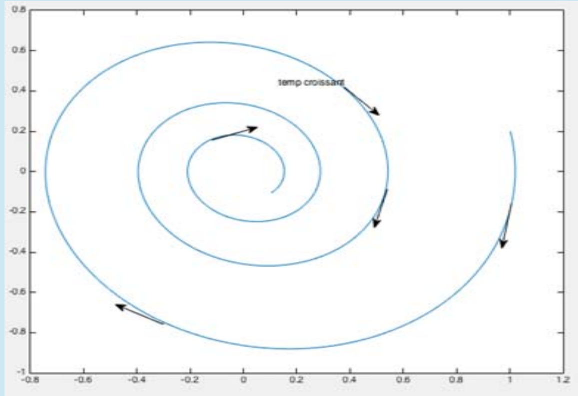
Den Hartog criterion

Derivatives of f about $\dot{z} = 0,\ \alpha_a = \alpha$

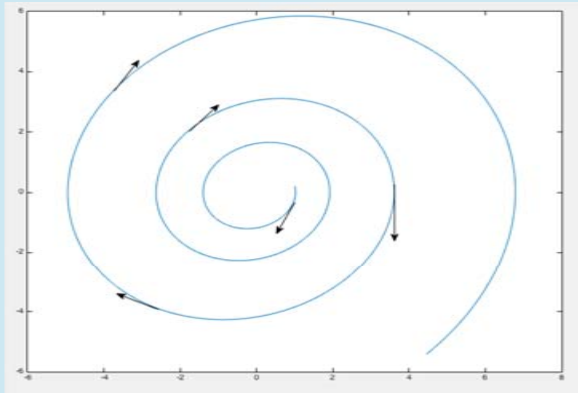
- In fact f only depends on $\dot{z}$, not on z ;
- $\alpha_a \simeq \alpha - \frac{\dot{z}}{V},\ \frac{\partial \alpha_a}{\partial \dot{z}} = -\frac{1}{V},\ ||V_a||^2 \simeq V^2$;
- $-\frac{\partial f}{\partial \dot{z}}(0,0) \simeq D = \frac{\varrho S V}{2}[c_x(\alpha) + \frac{\partial c_z}{\partial \alpha}(\alpha)]$;
- $\frac{\partial f}{\partial z}(0,0) = 0$;
- The linearized model is :

$$\boxed{\begin{array}{l} M\ddot{z} + D\dot{z} + Kz = f(0,0), \\ z(0) = \dot{z}(0) = 0. \end{array}}$$

Solutions pour $D > 0$



Solutions pour $D < 0$



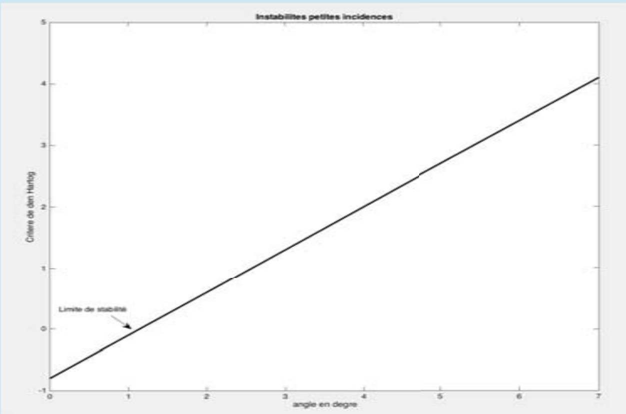
D is the aerodynamical damping.

Den Hartog criterion (next)

Retour

$D(\alpha)$ For Tacoma (H cross section)

$$\boxed{D = \frac{\varrho SV}{2}[c_x(\alpha) + \frac{\partial c_z}{\partial \alpha}(\alpha)]}$$

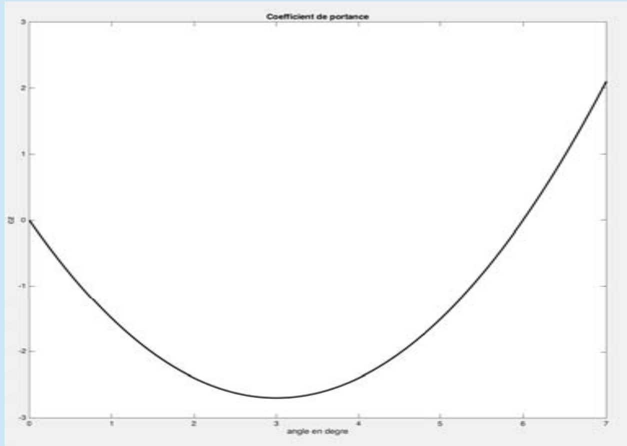


Den Hartog criterion : Zoom ↑;

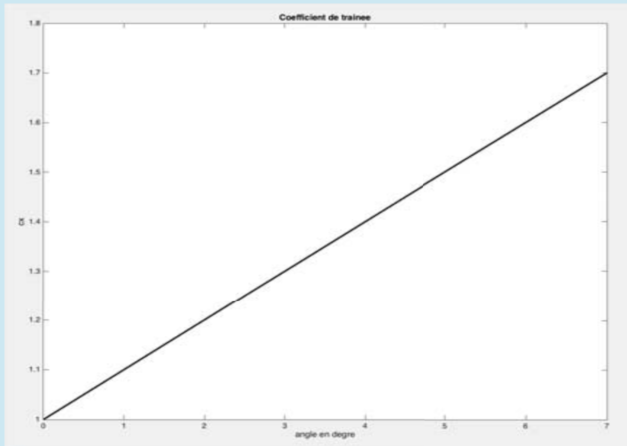
$$\boxed{D < 0 \;\; => \;\; \text{unstable}, \;\; D > 0 \;\; => \;\; \text{stable.}}$$

Remark One should take into account the structural damping.

c_z

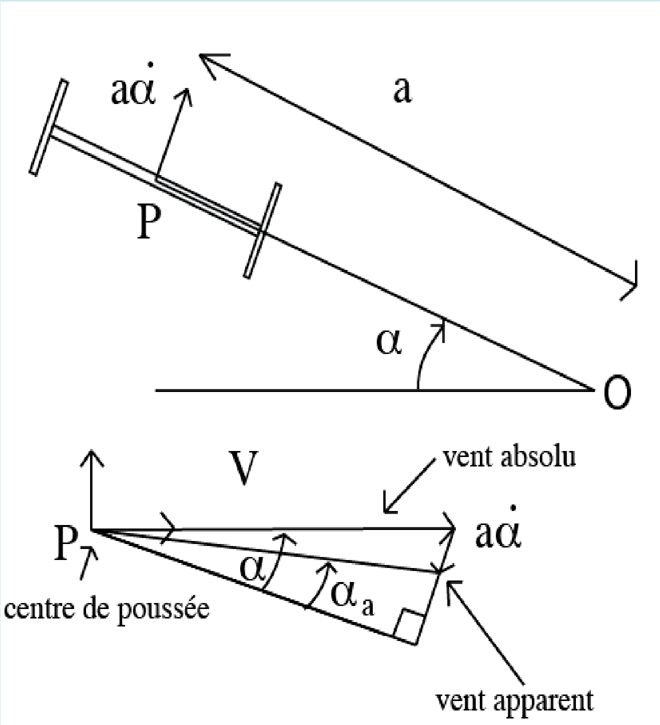


c_x



The pitching movement

The pitching movement



The movement of a cross section is represented by a rotation around point O.

Apparent wind

- The aerodynamic center is denoted P;
- the distance between O and P is a;
- $V_a = Ve_1 - \dot{\alpha}a[\sin(\alpha)e_1 + \cos(\alpha)e_2]$;
- $\alpha_a = \arctan(\tan(\alpha) - \frac{a\dot{\alpha}}{V\cos(\alpha)})$;
- $R(\alpha, \dot{\alpha}) = \frac{\varrho SL||V_a||^2}{2} c_{m_0}(\alpha_a)$.

The pitching model

$$J_0\ddot{\alpha} + C(\alpha - \alpha_0) = R(\alpha, \dot{\alpha}),$$

$$\alpha(0) = \dot{\alpha}(0) = 0.$$

Scanlan criterion

Linearisation

- Let us notice that : $\dfrac{\partial \alpha_a}{\partial \dot{\alpha}}(\alpha_0) = -\dfrac{a\cos(\alpha_0)}{V}$;
- $||V_a||^2 = V^2(1 - 2\dfrac{\dot{\alpha}a}{V}\sin(\alpha) + (\dfrac{\dot{\alpha}a}{V})^2) \simeq V^2(1 - 2\dfrac{\dot{\alpha}a}{V}\sin(\alpha_0)\ldots$;
- $c_{m_0}(\alpha_a) = c_{m_0}(\alpha_0) + \dfrac{\partial c_{m_0}}{\partial \alpha}(\alpha_0)\dfrac{\partial \alpha_a}{\partial \dot{\alpha}} + \ldots$

Linearized model around α_0 et $\alpha(0) = \dot{\alpha}(0) = 0$

$$J_0\ddot{\alpha} + D\dot{\alpha} + [C - \frac{\varrho SLV^2}{2}\frac{\partial c_{m0}}{\partial \alpha}(\alpha_0)](\alpha - \alpha_0) = \frac{\varrho SLV^2}{2}c_{m_0}(\alpha_0).$$

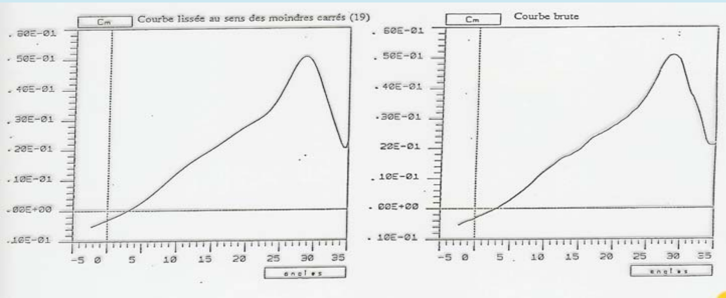
$$\boxed{D = a\frac{\varrho SLV}{2}[2c_{m_0}(\alpha_0)\sin(\alpha_0) + \frac{\partial c_{m_0}}{\partial \alpha}(\alpha_0)\cos(\alpha_0)].}$$

is the aerodynamical damping $=>$ Scanlan criterion :

$\boxed{\text{If } D < 0 \text{ the bridge is unstable and stable if } D > 0.}$

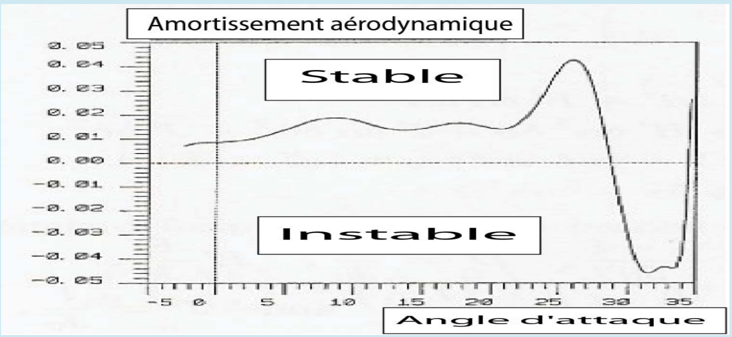
A military aircraft in our wind tunnel Rafale

Pitching coefficient c_{m_0}



Decrease of C_{m_0} about 28^0.

Damping curve *D*

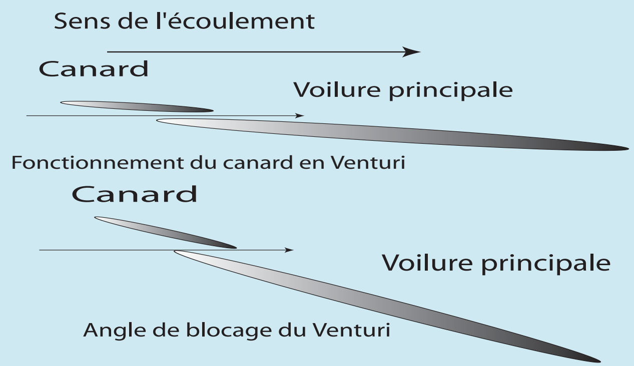


The instability has been checked in the tests (twice !).

An explanation

The *canard* behaves as a Venturi. But it doesn't work for an angle of attack about 28^0. This implies a sudden backward movement of the aerodynamic center.

The pitching moment is therefore quickly changing about 28^0

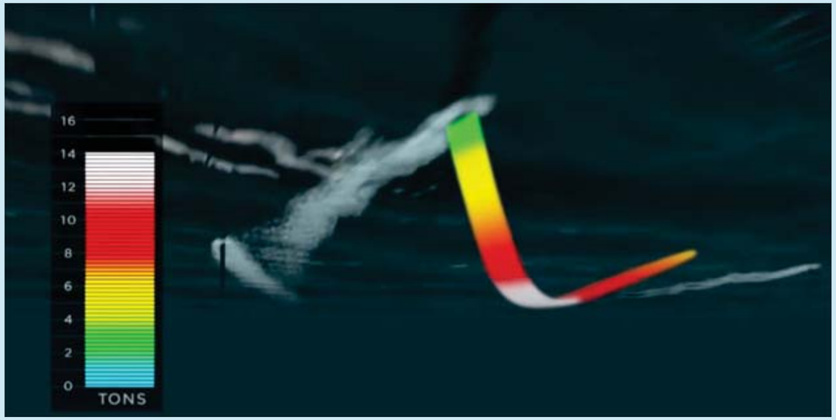


Functions of the foil

The article from OTUSA

The foil (L shape)

Two forces : the wind (propulsion) and the water (lift, drag, anti-drift).



How the foil is moved

As on old aircrafts, with hydraulic !

The slamming !

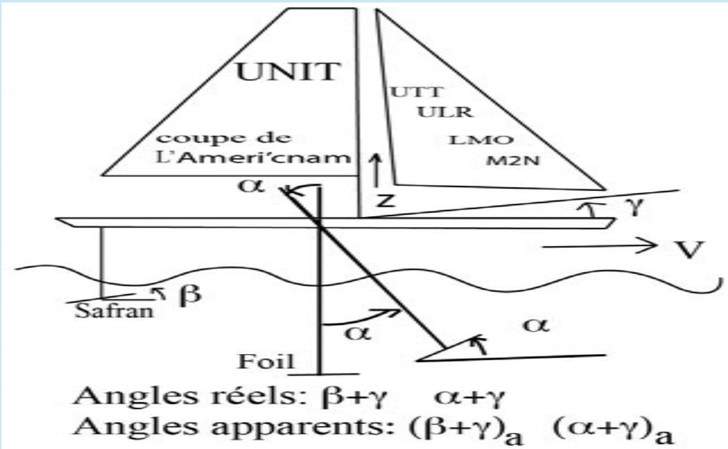
One has to avoid the slamming which is catatrophic. The foil has several functions : lift, anti-drift and control through the rake angle.

New Zealand 2017

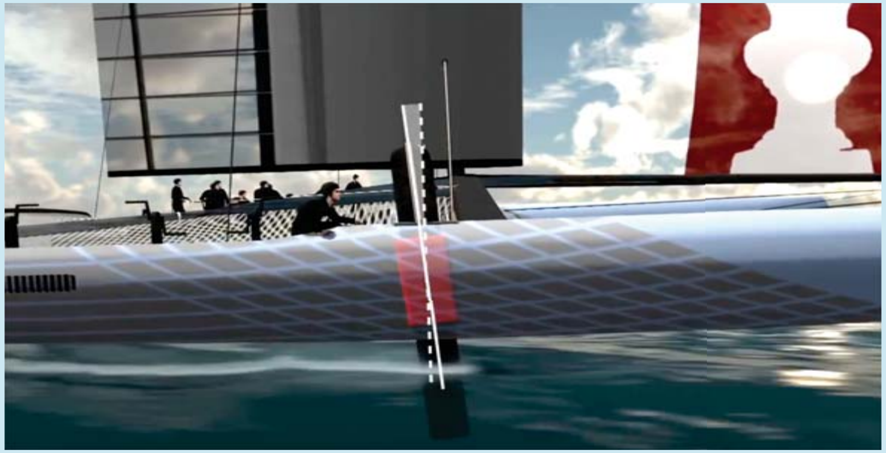
The mathematical model used

The basic ideas

Zoom↓



The scheme *in situ*



2 DOF : pitch is γ, heave at O is z ; β is fixed, α is the control

(G is behind O if $a > 0$ and before if $a < 0$; the moment are computed at O.)

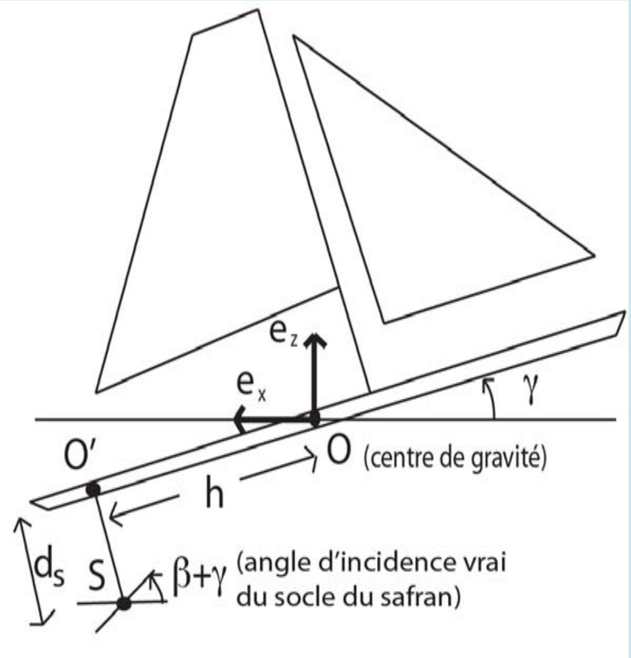
$$M\ddot{z} - aM\cos(\gamma)\ddot{\gamma} = -Mg + \frac{\varrho S_s|V_{as}|^2}{2} c_{zs}((\beta+\gamma)_a) + \frac{\varrho S_f|V_{af}|^2}{2} c_{zf}((\alpha+\gamma)_a),$$

$$-aM\cos(\gamma)\ddot{z} + J_0\ddot{\gamma} = -M_0 + \frac{\varrho S_s L|V_{as}|^2}{2} c_{ms}((\beta+\gamma)_a) + \frac{\varrho S_f L|V_{af}|^2}{2} c_{mf}((\alpha+\gamma)_a) + \frac{\varrho}{2}$$

$$\left[S_f d_f \sin(\alpha+\gamma)|V_{af}|^2 c_{zf}((\alpha+\gamma)_a) - S_s(h\cos(\gamma) - d_s\sin(\gamma))|V_{as}|^2 c_{zs}((\beta+\gamma)_a)\right]$$

Apparent velocity of the rudder

Angles β and γ



Zoom↑

Velocity of point S

(z is the heaving of O)

$$V_S = (\dot z - \dot\gamma(h\cos(\gamma) - d_S\sin(\gamma))e_Z - \dot\gamma(h\sin(\gamma) + d_S\cos(\gamma))e_X$$

Apparent velocity at S

$$V_{as} = (V + \dot\gamma(h\sin(\gamma) + d_S\cos(\gamma)))e_X - (\dot z - \dot\gamma(h\cos(\gamma) - d_S\sin(\gamma)))e_Z$$

Modulus of the apparent velocity at S

$$|V_{as}|^2 = (V + \dot\gamma(h\sin(\gamma) + d_S\cos(\gamma)))^2 + (\dot z - \dot\gamma(h\cos(\gamma) - d_S\sin(\gamma)))^2$$

Apparent pitching angle at S

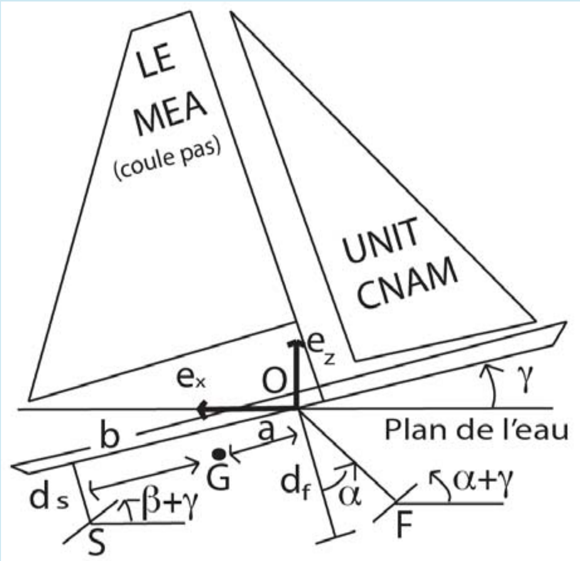
$$(\beta+\gamma)_a = \arcsin\left(\frac{[V_{as} \wedge (\cos(\beta+\gamma)e_X - \sin(\beta+\gamma)e_Z].e_Y}{|V_{as}|}\right)$$

$$OS = (h\cos(\gamma) - d_s\sin(\gamma))e_x - (h\sin(\gamma) + d_s\cos(\gamma))e_z$$

NB : The hydrodynamic center of the foil is assumed to be at S.

Apparent velocity for the foil

Velocity and pitching angle of the foil



Zoom ↑ $h = a+b = |O'O|$

Velocity of point F
(z is the heaving at O)

$$\boxed{V_f = (\dot z + \dot\gamma d_f \sin(\gamma + \alpha))e_z - \dot\gamma d_f \cos(\gamma + \alpha)e_x}$$

Apparent velocity at F

$$\boxed{V_{af} = (V + \dot\gamma d_f \cos(\gamma+\alpha)))e_x - (\dot z + \dot\gamma(d_f \sin(\gamma+\alpha)))e_z}$$

Modulus of the apparent velocity at F

$$\boxed{|V_{af}|^2 = (V + \dot\gamma d_f \cos(\gamma+\alpha))^2 + (\dot z + \dot\gamma d_f \sin(\gamma+\alpha))^2}$$

Apparent pitching angle at F

$$\boxed{(\alpha + \gamma)_a = \text{arsin}(\frac{[V_{af} \wedge (\cos(\alpha+\gamma)e_x - \sin(\alpha+\gamma)e_z].e_y}{|V_{af}|})}$$

$$\boxed{OF = -d_f \sin(\alpha+\gamma)e_x - d_f \cos(\alpha+\gamma)e_z}$$

NB : The hydrodynamic center is assumed to be at F**.**

How to simplify

A (simple) computation enables to linearize around a given position : $\gamma = 0,\ \beta = \beta_0,\ \alpha = \alpha_0$ ($\delta = \alpha - \alpha_0$) is the control angle (*the rake*) :

$$\boxed{\begin{array}{l} M\ddot{z} - aM\ddot{\gamma} = F_1 + T_1\gamma + C_{11}\dot{z} + C_{12}\dot{\gamma} + B_1\delta + E_1\dot{\delta}, \\ -aM\ddot{z} + J_0\ddot{\gamma} = F_2 + T_2\gamma + C_{21}\dot{z} + C_{22}\dot{\gamma} + B_2\delta + E_2\dot{\delta}. \end{array}}$$

with : Expressions of C_{ij} Expressions of B_i et E_i Expressions of $R_i = T_i/(\varrho V^2)$

$$F_1 = -Mg + \frac{\varrho V^2}{2}(S_s c_{zs}(\beta_0) + S_f c_{zf}(\alpha_0)), \qquad B_1 = \frac{\varrho S_f}{2}\frac{\partial(|V_{af}|^2 c_{zf}((\alpha+\gamma)a))}{\partial\gamma}$$

$$F_2 = -M_0 + \frac{\varrho L V^2}{2}(S_s c_{ms}(\beta_0) + S_f c_{mf}(\alpha_0)), \qquad B_2 = \frac{\varrho S_f}{2}L\frac{\partial(|V_{af}|^2 c_{mf}((\alpha+\gamma)a))}{\partial\gamma},$$

$$C_{11} = \frac{\varrho}{2}\frac{\partial(S_s|V_{as}|^2 c_{zs}((\beta+\gamma)a) + S_f|V_{af}|^2 c_{zf}((\alpha+\gamma)a))}{\partial\dot{z}}, \qquad E_1 = \frac{\varrho S_f}{2}\frac{\partial(|V_{af}|^2 c_{zf}((\alpha+\gamma)a))}{\partial\dot{\gamma}}$$

$$C_{12} = \frac{\varrho}{2}\frac{\partial(S_s|V_{as}|^2 c_{zs}((\beta+\gamma)a) + S_f|V_{af}|^2 c_{zf}((\alpha+\gamma)a))}{\partial\dot{\gamma}}, \qquad E_2 = \frac{\varrho S_f}{2}\frac{\partial(|V_{af}|^2 c_{mf}((\alpha+\gamma)a))}{\partial\dot{\gamma}}$$

$$C_{21} = \frac{\varrho}{2}\frac{\partial(S_s L|V_{as}|^2 c_{ms}((\beta+\gamma)a) + S_f L|V_{af}|^2 c_{mf}((\alpha+\gamma)a))}{\partial\dot{z}}, \qquad T_1 \text{ as } C_{12} \text{ but derivated } /\gamma,$$

$$C_{22} = \frac{\varrho}{2}\frac{\partial(S_s L|V_{as}|^2 c_{ms}((\beta+\gamma)a) + S_f L|V_{af}|^2 c_{mf}((\alpha+\gamma)a))}{\partial\dot{\gamma}}, \qquad T_2 \text{ as } C_{22} \text{ but derivated } /\gamma,$$

these derivatives are computed for : $z = \dot{z} = \gamma = \dot{\gamma} = 0,\ \alpha = \alpha_0;\ \beta = \beta_0$, being fixed.

The equilibrium configuration

First of all, let us define (β_0, α_0) which is the equilibrium position $(\gamma_0 = z_0 = 0)$ for a velocity V by : (M_0 is the external moment at O) :

$$\boxed{\begin{array}{l} S_s c_{zs}(\beta_0) + S_f c_{zf}(\alpha_0) = \dfrac{2Mg}{\varrho V^2}, \\[2ex] -\dfrac{S_s h}{L} c_{zs}(\beta_0) + \dfrac{S_f d_f \sin(\alpha_0)}{L} c_{zf}(\alpha_0) + S_s c_{ms}(\beta_0) + S_f c_{mf}(\alpha_0) = \dfrac{2M_0}{\varrho L V^2}. \end{array}}$$

Then we linearize the model around $\alpha = \alpha_0, \ \gamma = 0$ et $\dot{\gamma} = \dot{z} = 0$. Let us set : $\delta = \alpha - \alpha_0$ which is the control variable (the rake). Let us point out that the control δ is assumed to be precribed once it has been computed, which implies real time implementation.

The control variable δ could be (and was) adjusted through hydraulic cylinders driven from the steering wheel of the helmsman.
See J. Spithill in 2013 at SF

Stability : episode 1

Let us first neglect the apparent velocity (static stability). The discussion rests on the real parts of the imaginary part of $\mu = \sqrt{\lambda}$ solution of :

$$\det \begin{vmatrix} -\mu^2 M, & \mu^2 aM - \varrho V^2 R_1 \\ \mu^2 aM, & -\mu^2 J_0 - \varrho V^2 R_2 \end{vmatrix} = \lambda(\lambda J_G + \varrho V^2 M(R_2 + aR_1)) = 0.$$

The value $\lambda = 0$ corresponds to a heaving movement which is controlled directly by the boat velocity V.

The other one, say : $\lambda = -V\sqrt{\varrho M}\sqrt{\frac{R_2+aR_1}{J_G}}$ $=> \mu = \pm i\sqrt{-\lambda}$- leads to stability if : $R_2 + aR_1 > 0$. Otherwise an instability occurs. As a is small, R_2 has (essentially) the sign of $\frac{\partial c_{mf}}{\partial \alpha}(\alpha_0) - \frac{\partial c_{ms}}{\partial \alpha}(\beta_0)$.

In conclusion of this first analysis one can say that the angle α_0 should equilibrate the angle β_0 through the derivatives of the pitching moments. It is worth to notice that (approximately) this choice doesn't depend on the velocity V of the boat but only on the derivatives of the hydrodynamic coefficients .

Few results :

Gravity center near the foil $a = .2h = 5$ m/s ... backwards $a = .8h = 5$ m/s

Stability : episode 2

We are now concerned by the dynamic stability. Hence we take into account the damping matrix $-C$ (If it has eigenvalue with negative real part one can have a stall flutter). The new equation to study for the stability is :

$$\det \begin{vmatrix} -\mu^2 M - i\mu C_{11} & \mu^2 aM - i\mu C_{12} - \varrho V^2 R_1 \\ \mu^2 aM - i\mu C_{21} & -\mu^2 J_0 - i\mu C_{22} - \varrho V^2 R_2 \end{vmatrix} =$$

$$\mu^4 J_G + i\mu^3 \left(\frac{J_0}{M} C_{11} + C_{22} + a(C_{12} + C_{21})\right) + \mu^2 \left(\varrho V^2 (R_2 + aR_1) - \frac{C_{11}C_{22} - C_{12}C_{21}}{M}\right)$$

$$+ i\mu \frac{\varrho V^2}{M} (R_2 C_{11} - R_1 C_{21}) = 0$$

One solution is still $\mu = 0$, but there are three other solutions. Due to the complexity of the expressions, we use a computational method. Let us see few results connected to the dynamic stability discussion :

Gravity center near the foil $a = .2 \;\; h = 5$ ... backwards $a = 1 \;\; h = 5$

On the stability of racing sailing boats with foils, Chin. Ann. Math., 2018

Stabilisation and control

$\forall \varepsilon > 0$, let us introduce a control criterion :

$$J^{\varepsilon}(\delta) = \frac{1}{2}||X(T)||^2 + \frac{1}{2}||\dot{X}(T)||^2 + \frac{\varepsilon}{2}\int_0^T [a_0\delta^2(s) + b_0\dot{\delta}^2(s)]ds,$$

where :

$$X = \begin{pmatrix} z \\ \\ \gamma \end{pmatrix}, \; X_0 = \begin{pmatrix} \delta z_0 \\ \\ \delta\gamma_0 \end{pmatrix}, \; X_1 = \begin{pmatrix} \delta z_1 \\ \\ \delta\gamma_1 \end{pmatrix}$$

and X is solution of (X_0 and X_1 are initial perturbations due for instance to the waves; the matrices $\mathcal{M}, \mathcal{K}, \mathcal{C}$ and the vectors $\mathcal{B}, \mathcal{E}$ are *self explanatory* :

$$\boxed{\mathcal{M}\ddot{X} - \mathcal{C}\dot{X} + \mathcal{K}X = \mathcal{B}\delta + \mathcal{E}\dot{\delta}, \; X(0) = X_0, \; \dot{X}(0) = X_1.}$$

The optimal control problem consists in finding $\delta \in H_0^1(]0, T[)$ which minimizes J^{ε} for a given initial perturbation.
On the Controllability of Racing Sailing Boats with Foils, AIMS's journ. 2018

Exact control, in fact phase control ($\simeq$ Tychonov)

Let us define the control $\delta^0 \in H_0^1(]0,T[)$ by :

$$\boxed{\begin{array}{l} a_0\delta^0 - b_0\ddot{\delta}^0 = (\mathcal{E}.\dot{P}^1)_2 - (\mathcal{B}.P^1)_2, \;\; \mathcal{M}\ddot{P}^1 + {}^t\mathcal{C}\dot{P}^1 + {}^t\mathcal{K}P^1 = 0, \\ \\ P^1(0) = \Phi_0, \;\; \dot{P}^1(0) = \Phi_1, \;\; \Phi = (\Phi_0, \Phi_1) \in \mathbb{R}^2, \end{array}}$$

where :

$$\boxed{\forall \delta\Phi = (\delta\Phi_0, \delta\Phi_1) \in \mathbb{R}^2, \;\; \Lambda(\Phi, \delta\Phi) = L(\delta\Phi).}$$

and the bilinear form Λ and linear L defined by :

$$\boxed{\begin{array}{l} \Lambda(\Phi,\delta\Phi) = \frac{2}{T}\sum_{n\geq 1} \dfrac{\int_0^T \xi(s)\sin(\frac{n\pi s}{T})ds \int_0^T \upsilon(s)\sin(\frac{n\pi s}{T})ds}{a_0 + b_0\frac{n^2\pi^2}{T^2}} \\ L(\delta\Phi) = (\mathcal{M}\dot{X}(0) - \mathcal{C}X(0), \delta\Phi_0)_2 - (\mathcal{M}X(0), \delta\Phi_1)_2 \end{array}}$$

with the notations (controllability is equivalent to : $P^0 = 0$) :

$$\left\{ \begin{array}{l} \mathcal{M}\ddot{Q} + {}^t\mathcal{C}\dot{Q} + {}^t\mathcal{K}Q = 0, \;\; Q(0) = \delta\Phi_0, \;\; \dot{Q}(0) = \delta\Phi_1, \\ \xi(s) = (\mathcal{B}.P^1)_2(s) - (\mathcal{E}.\dot{P}^1)_2(s), \quad \upsilon(s) = (\mathcal{B}.Q)_2(s) - (\mathcal{E}.\dot{Q})_2(s), \end{array} \right.$$

Remark concerning controllability

It seems a good idea to restrict the control to the term : $\mathcal{B}\delta$ if $\mathcal{E}$ est negligible because the model is simplier. But $-\mathcal{B}$ also a part of the second column of the stiffnes matrix $\mathcal{K}$ (the first one is zero). Let us set : $\mathcal{K}2 = -\mathcal{B} - \mathcal{G}$.

The controllability is not so obvious Proof in the general case ; . Let us consider the case : $-\mathcal{C}$ (damping matrix) is negligible (*no apparent flow*). The control criterion leads to : Proof if $\mathcal{C} = 0$

$$\boxed{(\mathcal{M}^{-1}\,{}^{t}\mathcal{K}\mathcal{D}.\mathcal{B})_2 = -(\mathcal{M}^{-1}\begin{pmatrix} (\mathcal{G}.\mathcal{D})_2 \\ 0 \end{pmatrix}.\mathcal{B})_2 \neq 0 \;\; \text{where } (\mathcal{D}.\mathcal{B})_2 = 0.}$$

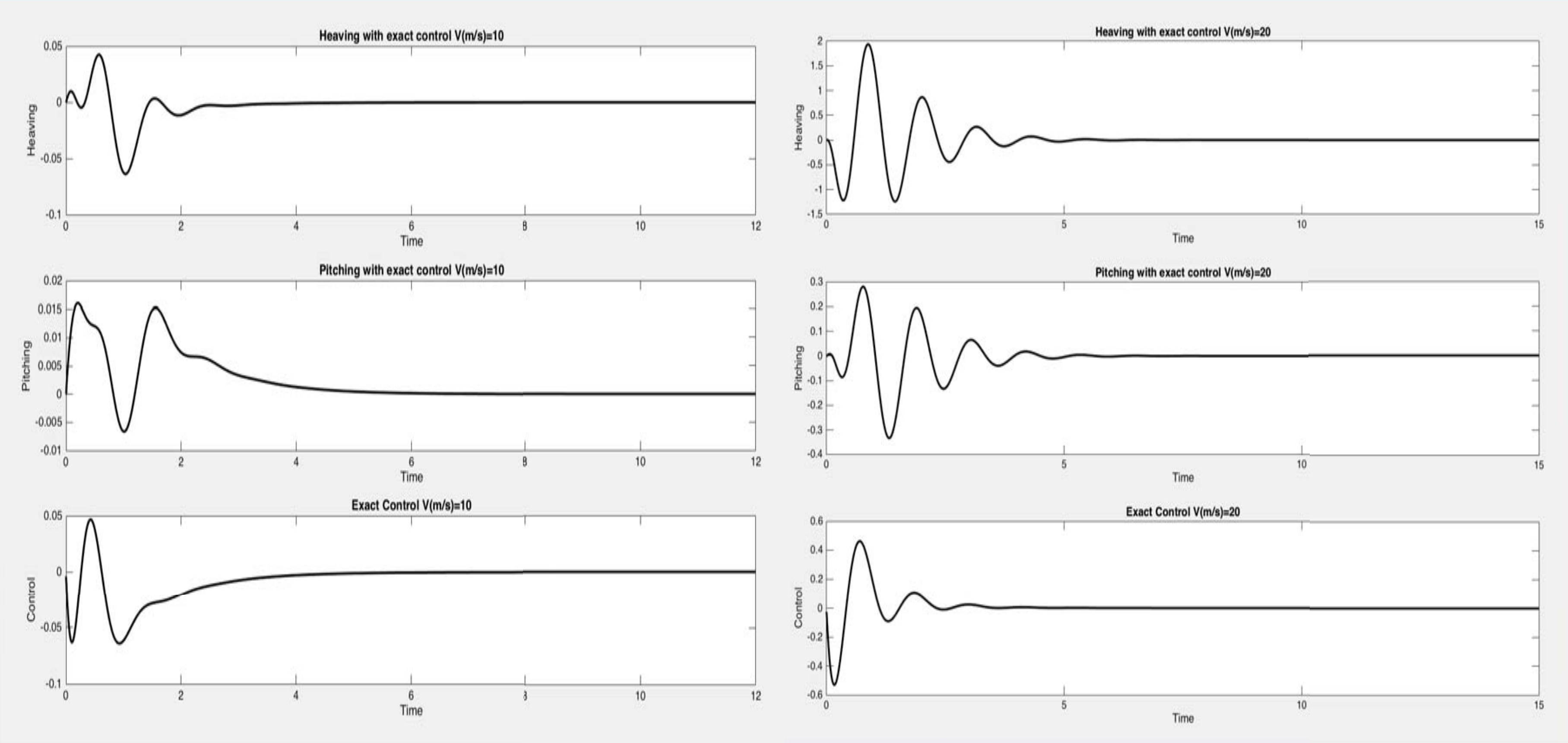
$$\text{or else : } (\mathcal{G}.\mathcal{D})_2 \neq 0 \textbf{ and } (\begin{pmatrix} J_0 \\ aM \end{pmatrix}.\mathcal{B})_2 \neq 0.$$

In fact it seems more securising to use the term $\mathcal{E}\dot{\delta}$. Furthermore, it enables one to avoid stiff transition on the rake by the possibility to prescribe $\delta(0) = \delta(T) = 0$.

Numerical tests

In the following we give few results obtained with an imaginary boat. An exact control in $H_0^1(]0, T[)$ has been used. The velocity of the boat is V (10 m/s and 20 m/s). We introduced an initial perturbation in each case.

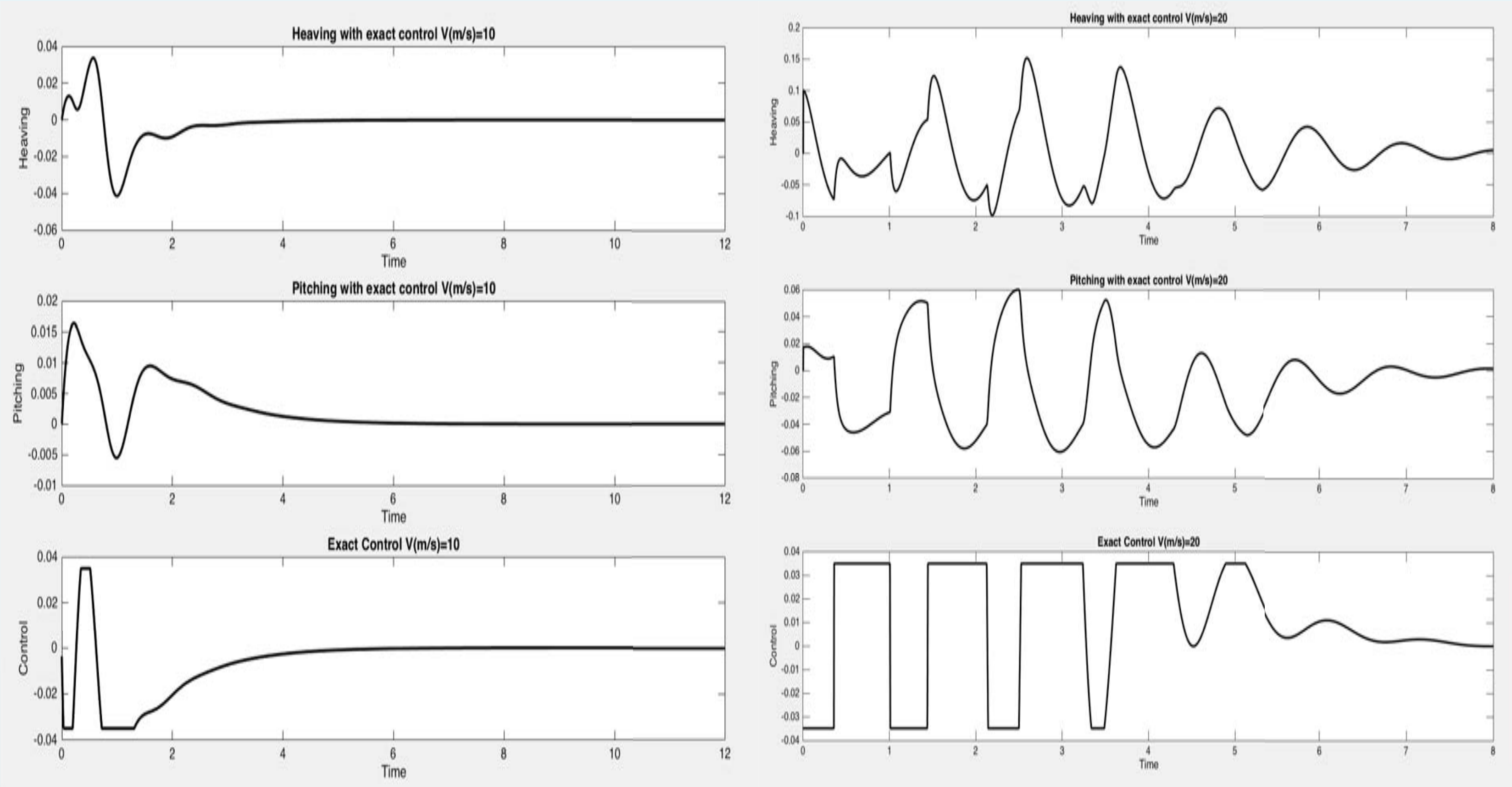
Zoom1 Zoom2



Bounded control

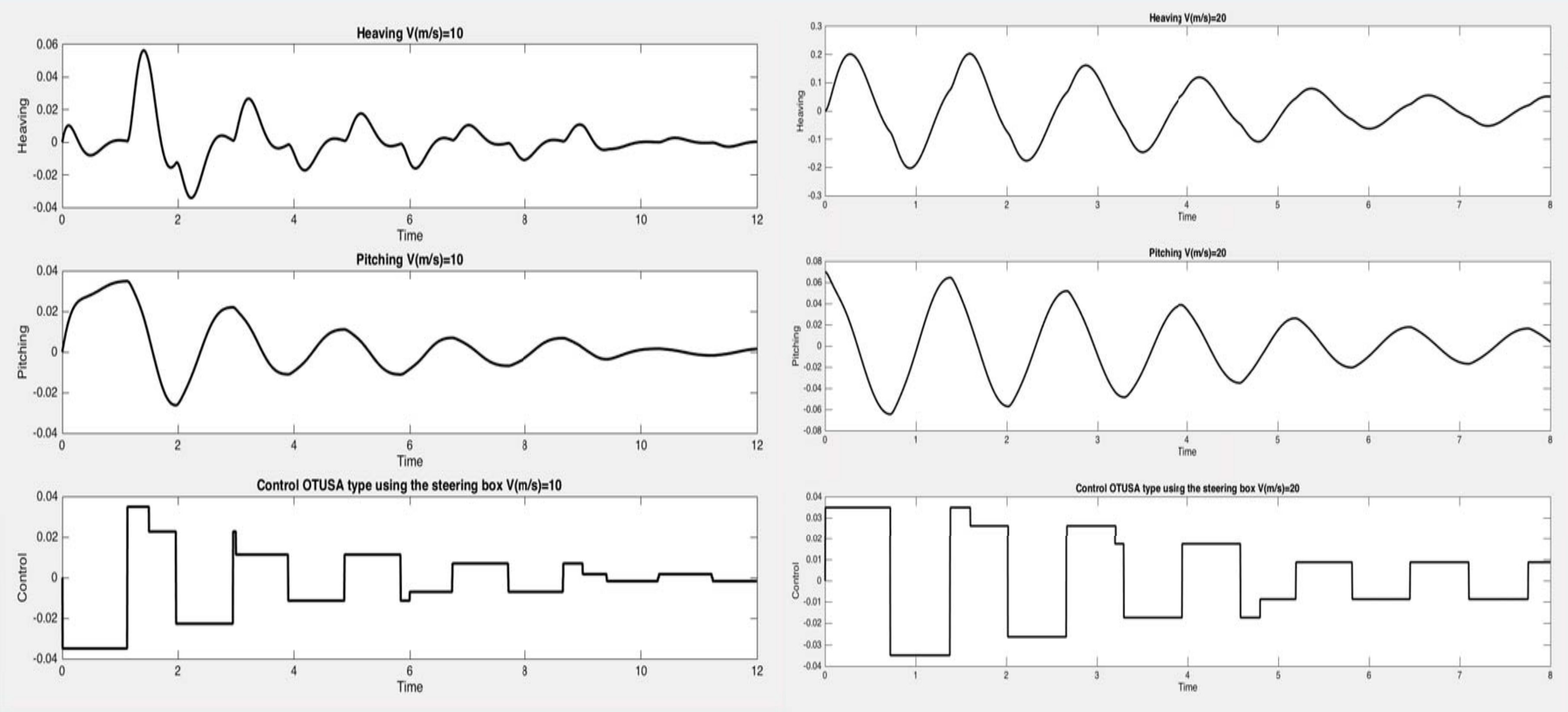
In this simulation we applied a bound on the control. Clearly the time necessary is longer, but the results are more physically acceptable.

Zoom1 Zoom2



Damping feedback

We use a *bang-bang control law* based on the sign of the pitching velocity the control is $\delta = -\mathrm{sign}(\dot{\gamma})\delta_{max}$. The results are more realistic than in the former cae, but the control is a little bit less efficient. The boat is the same, the velocities are identical and the bound of the control δ_{max} is also the same, but decreasing as Oracle did in 2013 : Oracle 2013 Zoom1 Zoom2 AC-45



Conclusions for the flying sailing boats

- A simple equation can be used for a first modelling of the boat behaviour ;
- The corner point is to generate polar curves for many configurations in quasi-static ;
- The dymamical model is built using the apparent velocity at the aero-hydro-dynamic- center of each structural component (sail, foil, rudder) ;
- The automatic control is well define from the linearized model as soon it is stable and preferably using both the *rake* and its time derivative ;
- For more complex movement of the foil (inverse T shape, wait a minute please...) Projet New Zealand 2021 AC 2021... Controls...

Thank you very much for your attention

Conclusions for the flying sailing boats

- A simple equation can be used for a first modelling of the boat behaviour ;
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Thank you very much for your attention

Lecture 2: Basic tools

Henri Poincaré and Ivar Bendixson

CONTENTS

Few recalls on ODEs

Voice can be activated from the bottom bar

Let us consider the following equation:

$$M\ddot{X} + C\dot{X} + KX = F(X, \dot{X}),\ X(0) = X_0,\ \dot{X}(0) = X_1, X(t) \in \mathbb{R}^N$$

In this lecture we assume that:

$$F \in \mathcal{C}^1(\mathbb{R}^{2N}; \mathbb{R}^N).$$

Theorem (Cartan existence and uniqueness of a solution)

The previous equation has a unique solution for any finite time t (this is a basic property in the phase diagram analysis).

Corollary (An simple but important consequence)

The trajectory in the space $(X(t), \dot{X}(t)) \in \mathbb{R}^{2N}$ can't have a double point for a finite time.

Phase diagram

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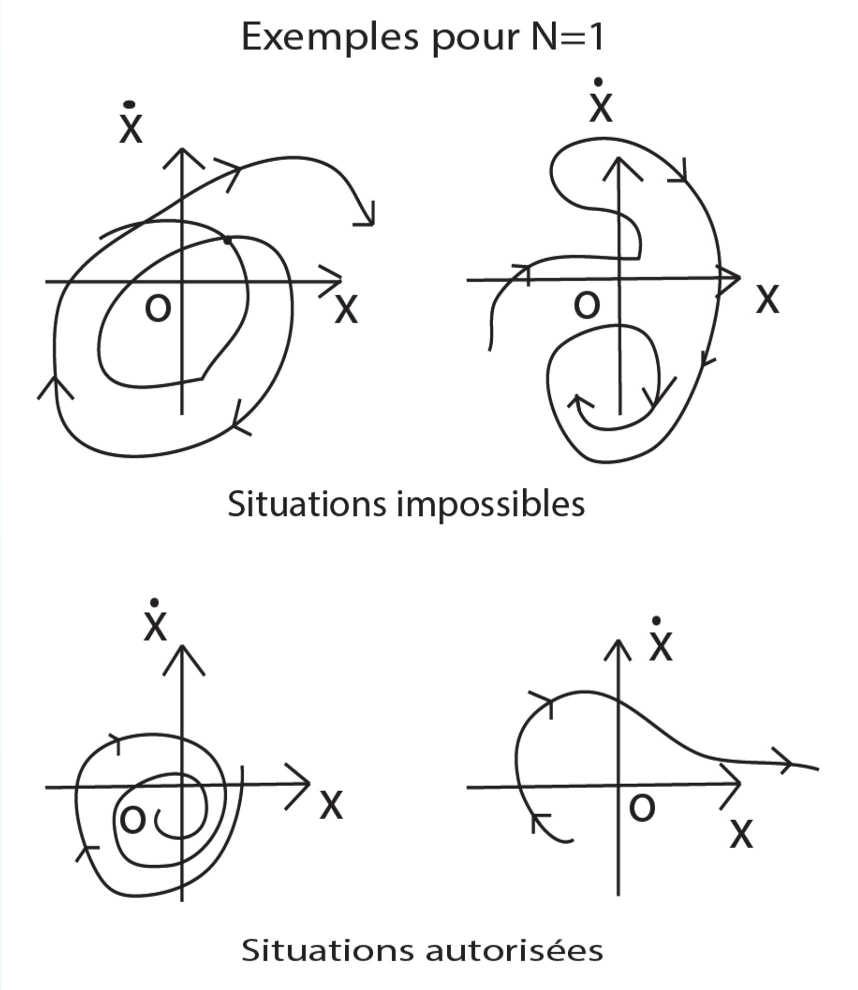

Cartan's Theorem implies practical consequences in order to localize the trajectories.

- No double points on a trajectory;
- If $\dot{x} > 0 \implies x$ is increasing;
- One can have an asymptotic behaviour above the axis x;

Limit cycle of oscillations

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Definition (Limit cycle of oscillations $M\ddot{X} + C\dot{X} + KX = F(X, \dot{X})$)

Let $X(t),\ t > 0$ be a solution of the previous (without initial conditions): $X(t),\ t > 0$ is limit cycle of oscillations if $\exists T \in \mathbb{R}^{+*}$ (the smallest value of T is the period), such that :

$$\boxed{\forall t \in \mathbb{R}^{+},\ X(t+T) = X(t).}$$

In the space $\mathbb{R}^{2N}$ $t > 0 \to (X(t), \dot{X}(t))$ is a closed curve the described by the parameter $t \in [0, T[$. Furthermore if :
$\exists t_0 \in]0, \infty]$ such that: $\dot{X}(t_0) = \ddot{X}(t_0) = 0$, the limit cycle $(X(t),\ t > t_0)$ is an equilibrium point (not necessarily stable) solution of:

$$KX = F(X, 0).$$

Resonance

Visu Prog

With polar coordinates ($N = 1$):
$(x, \dot{x}) \to (r, \varphi)$
$\dot{r} = ar - br^3,\ \dot{\varphi} = \omega + cr^2$

Van der Pol

Visu Prog.

$N = 1,\ \ddot{x} + 2\omega\varepsilon\dot{x}(x^2 - 1) + \omega^2 x = 0$
$(0, 0)$ is an instable equilibrium.

Poincaré-Bendixson's criterion

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Let us consider the equation (there can be a damping included in f and $N = 1$):

$$m\ddot{x} + kx = f(x, \dot{x}), \; x(0) = x_0, \; \dot{x}(0) = x_1, \; x(t) \in \mathbb{R}^N.$$

Let us assume that $x(t), \; t > 0$ is a limit cycle of oscillations.
The curve representing $(x(t), \dot{x}(t)), \; t > 0$ is the boundary in the plane $x_1 = x, \; x_2 = \dot{x}$, of a connected open set D with boundary∂D and the unit outwards normal to D along ∂D is:

$$\nu = (\nu_1, \nu_2) = \frac{1}{\sqrt{\dot{x}^2 + \ddot{x}^2}}(-\ddot{x}, \dot{x}), \text{ one has } \partial D : \; ds = \sqrt{\dot{x}^2 + \ddot{x}^2} dt.$$

Let us set $P = (0, f(x, \dot{x}))$ et on a sur ∂D:

$$(P, \nu) ds = f(x, \dot{x})\dot{x} dt = (m\ddot{x} + kx)\dot{x} dt.$$

From **Stokes**'s formula Proof:

$$\boxed{\int_D div(P) dx d\dot{x} = \int_D \frac{\partial f}{\partial \dot{x}}(x, \dot{x}) dx d\dot{x} = \int_{\partial D} f(x, \dot{x}) \nu_2 ds = \int_0^T (m\ddot{x} + kx)\dot{x} dt = 0.}$$

Formulation of the criterion

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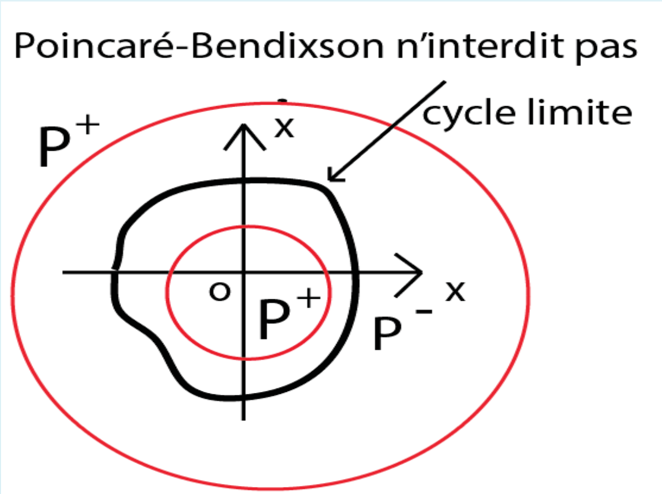
We introduce two sets (let us recall that $N = 1$):

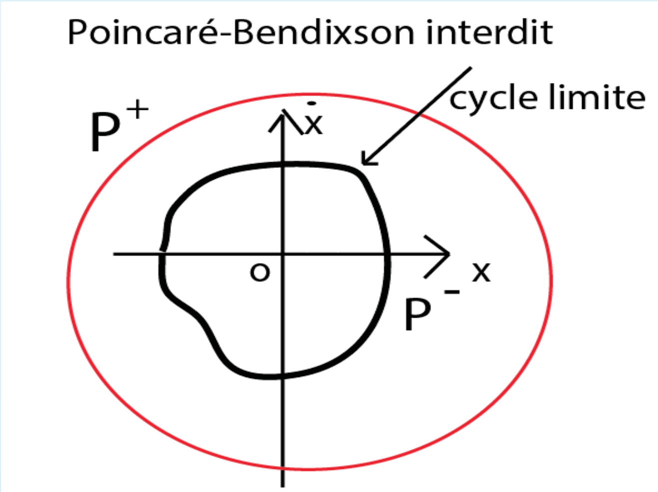
$$\boxed{P^+ = \{(x,\dot{x}) | \frac{\partial f}{\partial \dot{x}}(x,\dot{x}) > 0\}, P^- = \{(x,\dot{x}) | \frac{\partial f}{\partial \dot{x}}(x,\dot{x}) < 0\}.}$$

Let $x(t),\ t > 0$ a limit cycle of oscillations and let D the open set delimited by this cycle in the phase diagram $(x,\dot{x})$.

Lemma (Poincaré-Bendixson's criterion)

The open set D can't be included neither in P^+ nor P^-. Be careful: the criterion involves the open set D, not its boundary ∂D.





The energy criterion

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$x(t),\ t > 0$ is still a limit cyle of oscillations ∂D which delimitates the open set D; $(N = 1)$. Let us notice that: Proof:

$$0 = \int_0^T (m\ddot{x} + kx)\dot{x}dt = \int_0^T f(x,\dot{x})\dot{x}dt = \int_0^T \frac{f(x,\dot{x}) - f(x,0)}{\dot{x}}\dot{x}^2 dt.$$

Let us introduce two sets:

$$\boxed{E^+ = \{(x,\dot{x}) | \frac{f(x,\dot{x}) - f(x,0)}{\dot{x}} > 0\},\ E^- = \{(x,\dot{x}) | \frac{f(x,\dot{x}) - f(x,0)}{\dot{x}} < 0\}}$$

Lemma (The energy criterion (PhD-MTR))

The trajectory ∂D solution of the equation:

$$m\ddot{x} + kx = f(x,\dot{x}),$$

can't be included neither in E^+ nor E^-. This is why one could say that the energy criterion is more accurate tha the Poincaré-Bendixson's one.

Poincaré-Bendixson's Theorem

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Definition (Invariant set by an ODE)

Let $I \subset \mathbb{R}^2$ and $(x_0, x_1) \in I$. Let us denote by $x(t),\ t > 0$ the solution of:

$$m\ddot{x} + kx = f(x, \dot{x}),\ x(0) = x_0,\ \dot{x}(0) = x_1.$$

If $\forall (x_0, x_1) \in I \Rightarrow \forall t > 0,\ (x(t), \dot{x}(t)) \in I$ then I is an invariant set of the previous ODE.

Theorem (Existence of a limit cycle of oscillations)

Let us assume that there exists an invariant set $I \neq \emptyset$ for the ODE which is compact (closed and bounded). Then one has two possibilties: 1) $\exists$ a limit cycle in I, 2) $\exists$ an equilibrium point on the axis x which can be reached for $t \to \infty$

(3 vidéos) Proof V1 Proof V2 Preuve V3

even if it is quite simple, this proof requires some mathematical precisions

How to look for invariant sets

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Let us discuss few examples for the following ODE:
$f(x,\dot{x}) = p(x) + \dot{x}q(x)$, P, Q being primitives of p, q. One has: Proof

$$\frac{d}{dt}\left(m\frac{\dot{x}^2}{2} + k\frac{x^2}{2} - P\right) = \dot{x}^2 q(x),$$

$$\frac{d}{dt}\left(m\frac{\dot{x}^2}{2} + k\frac{x^2}{2} - P - \dot{x}Q + \frac{Q^2}{2m}\right) = \frac{Q}{m}(kx - p),$$

$$R^- = \{(x,\dot{x}) | q(x) \leq 0\} \text{ and } S^- = \{(x,\dot{x}) | Q(kx - p) \leq 0\}$$

Lemma (see the video for other examples (Other examples))

Let $\exists c_0 > 0,\ c_1 > 0$ s.t. the curves of the phase plan:

$$H = \{(x,\dot{x}) | (m\frac{\dot{x}^2}{2} + k\frac{x^2}{2} - P = c_0\}$$
$$J = \{(x,\dot{x}) | m\frac{\dot{x}^2}{2} + k\frac{x^2}{2} - P - \dot{x}Q + \frac{Q^2}{2m} = c_1\}.$$

are closed and included in R^- (resp. S^-). Then the set delimited by H (resp. J) is an invariant set for the ODE …

Few examples

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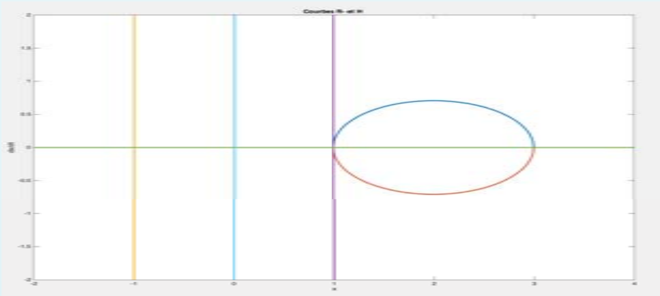
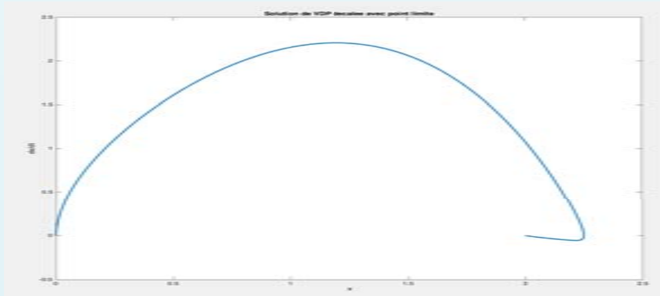
Let us set (Van der Pol type): Trajectory

$$q(x) = 1 - x^2 \Rightarrow Q(x) = x - \frac{x^3}{3} + c \text{ et } p(x) = 2k \Rightarrow P(x) = 2kx - 2k$$

One obtains for R^- (left) and the trajectory (right):

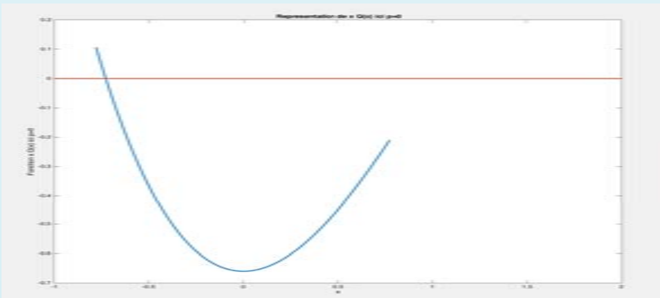


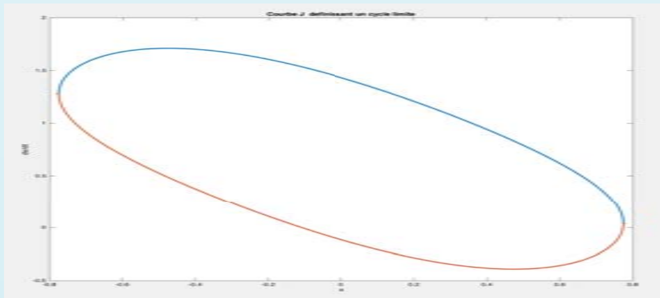


We set (Van der Pol): Building of J

$$q(x) = 1 - x^2 \Rightarrow Q(x) = x - \frac{x^3}{3} + c_0 \text{ et } p(x) = 0 \Rightarrow P(x) = 0$$

One obtains S^- (left) and trajectory J (right):





QCM

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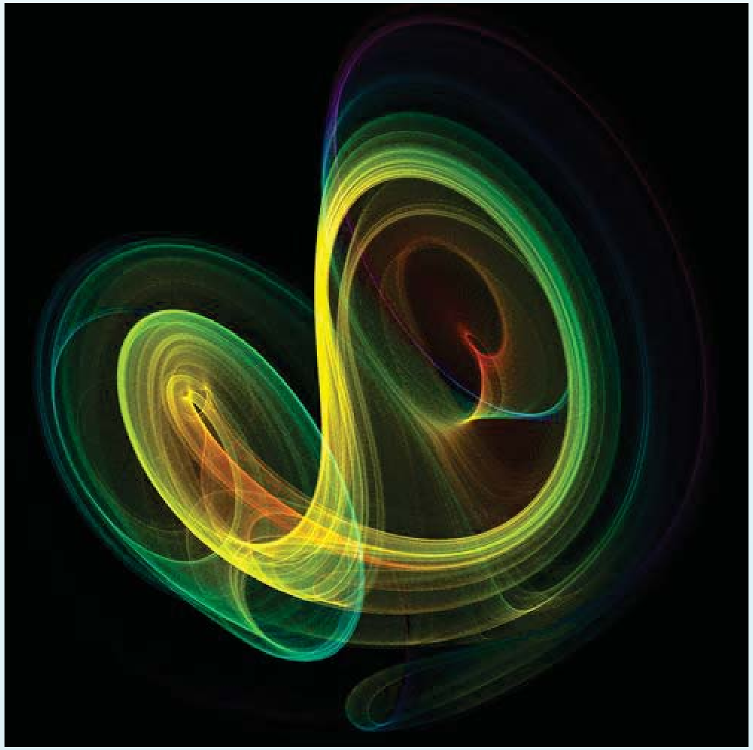


Trajectory for $N = 3$ for an ODE.

Answer to the questions and check your score.

Run the qcm

How to build the limit cycle

The voice is driven from the bottom bar.

Lecture 3: How to build the limit cycle

Limit cycles of oscillation are part of the nature

PLAN

1. The canonical form
   - First change of variables
   - Second change of variables
   - Third change of variables
2. Study of the resonant term
   - Expression of the resonant term
   - Signature of the resonant term
3. Algorithm
   - With symbolic computation...
   - Organisation of the program
4. QCM

First change of variables

The voice is driven from the bottom bar.

For sake of simplicity we restrict the description with the following simplified EDO :

Find $x(t)$ solution of :
$$\ddot{x} + \omega^2 x = f(x, \dot{x}), \;\; x(0) = x_0, \;\; \dot{x}(0) = x_2.$$

In the neighbourhood of $(0, 0)$:

$$g(x, \dot{x}) = f(x, \dot{x}) - f(0, 0) - x\frac{\partial f}{\partial x}(0, 0) - \dot{x}\frac{\partial f}{\partial \dot{x}}(0, 0)$$
$$=> |g(x, \dot{x})| = O(|x|^2 + |\dot{x}|^2),$$

and up to a translation with respect to x, one can assume that $f(0, 0) = 0$. The model becomes :

$$\boxed{\ddot{x} - \frac{\partial f}{\partial \dot{x}}(0, 0)\dot{x} + (\omega^2 - \frac{\partial f}{\partial x}(0, 0))x = g(x, \dot{x}).}$$

We consider the case where the linearized model is unstable.

First order differential system

The voice is driven from the bottom bar.

Let us set :

$$\boxed{X=\begin{pmatrix} x \\ \dot{x} \end{pmatrix},\ A=\begin{pmatrix} 0 & 1 \\ -(\omega^2-\frac{\partial f}{\partial x}(0,0)) & -\frac{\partial f}{\partial \dot{x}}(0,0) \end{pmatrix},\ B=\begin{pmatrix} 0 \\ g(x,\dot{x}) \end{pmatrix}.}$$

This model is equivalent to the following one :

$$\boxed{\frac{dX}{dt}=AX+B(X),\ X(0)=X_0=\begin{pmatrix} x_0 \\ x_1 \end{pmatrix}.}$$

The eigenvalues λ of A are solutions of :

$$\lambda^2-\lambda\frac{\partial f}{\partial \dot{x}}(0,0)+(\omega^2-\frac{\partial f}{\partial x}(0,0))=0.$$

They are assumed to be as follows :

$$\lambda = a \pm ib \text{ avec } a \geq 0 \text{ et } b \neq 0.$$

The instability considered corresponds to $a > 0$ but also : $a << 1$ which is the characteristic of a *stall flutter*.

Elimination of the coupling in the first order terms

The voice is driven from the bottom bar.

We introduce the first change of variables :

$$X = \begin{pmatrix} 1 & 1 \\ \lambda & \overline{\lambda} \end{pmatrix} Y = DY,$$

which leads to the model where $G(Y) = g(x, \dot{x})$:

$$\boxed{\dot{Y} = \begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix} Y + \frac{i}{2b} \begin{pmatrix} G(Y) \\ -G(Y) \end{pmatrix}.}$$

It is worth noting that the second equation is the complex conjugate of the first one because $G(Y) \in \mathbb{R}$.

Therefore we focus on the first one.
Let us point out that :

$$G_2(y_1, \overline{y}_1) = G(Y)$$

and

$$|G_2(y_1, \overline{y}_1)| = O(|y_1|^2 + |\overline{y}_1|^2).$$

$$\boxed{\dot{y}_1 = \lambda y_1 + \frac{i}{2b} G_2(y_1, \overline{y}_1)}$$

Second change of variables

The voice is driven from the bottom bar.

One can check that we can write :

$$\boxed{\begin{array}{l} G_2(y_1,\overline{y}_1) = d_{11}y_1^2 + d_{12}y_1\overline{y}_1 + d_{22}\overline{y_1}^2 + G_3(y_1,\overline{y}_1), \\ \text{avec : } |G_3(y_1,\overline{y}_1)| = O(|y_1|^3 + |\overline{y}_1|^2). \end{array}}$$

We introduce a change of variables in the vicinity of the origin in $\mathbb{C}$:

$$y_1 = z + p(z,\overline{z}) \text{ où } p \in \mathcal{P}_2 = \{\alpha z^2 + \beta z\overline{z} + \gamma\overline{z}^2\}.$$

De $\dot{y}_1 = (1 + \frac{\partial p}{\partial z})\dot{z} + \frac{\partial p}{\partial \overline{z}}\dot{\overline{z}}$, et $\dot{\overline{y_1}} = \overline{\lambda}\overline{y}_1 + O(|y_1|^2)$,

and from a simple computation we deduce that : Proof

$$\boxed{\begin{array}{l} \dot{z} = \lambda z + \lambda p - \lambda\frac{\partial p}{\partial z}z - \overline{\lambda}\frac{\partial p}{\partial \overline{z}}\overline{z} + d_{11}z^2 + d_{12}z\overline{z} + d_{22}\overline{z}^2 + H_3(z,\overline{z}) \\ \text{avec } H_3(z,\overline{z}) = O(|z|^3) \end{array}}$$

The challenge is to find the polynomial p in order to destroy the second order terms.

Elimination of the terms of order 2

The voice is driven from the bottom bar.

Let us notice that :

$$\boxed{\mathcal{L}(p) = \lambda p - \lambda \frac{\partial P}{\partial z} z - \overline{\lambda} \frac{\partial p}{\partial \overline{z}} \overline{z}}$$

is a linear mapping from $\mathcal{P}_2$ into itself. Its matrix representation in the basis : $\{z^2, z\overline{z}, \overline{z}^2\}$ is :

$$L = \begin{pmatrix} -\lambda & 0 & 0 \\ 0 & -\overline{\lambda} & 0 \\ 0 & 0 & \lambda - 2\overline{\lambda} \end{pmatrix} = \begin{pmatrix} -a - ib & 0 & 0 \\ 0 & -a + ib & 0 \\ 0 & 0 & -a + 2ib \end{pmatrix}.$$

Hence L is a one to one mapping in a vicinity of the origin in $\mathbb{C}$ as far a none of the diagonal terms are small. This is the case because $b \neq 0$.

Therefore all the terms of order 2 can be cancelled from this non linear change of variables.

Third change of variables

The voice is driven from the bottom bar.

Let us use again the same strategy as before but that time in order to try to cancel terms of order 3. Hence we set :

$$z = \xi + q(\xi, \overline{\xi}) \text{ où } q \in \mathcal{P}_3.$$

The same operator $\mathcal{L}$ appears but that time from $\mathcal{P}_3$ into itself. Its matrix in the basis $\{\xi^3, \xi^2\overline{\xi}, \xi\overline{\xi}^2, \overline{\xi}^3\}$ est :

$$L = \begin{pmatrix} -2(a-ib) & 0 & 0 & 0 \\ 0 & -2a & 0 & 0 \\ 0 & 0 & -2(a-ib) & 0 \\ 0 & 0 & 0 & -2a+4ib \end{pmatrix}.$$

One can see that the term $\xi^2\overline{\xi}$, which is an eigenvector of $\mathcal{L}$ is (almost) in the kernel because a is small. The other terms can be eliminated, ensuring that we remain in a neighbourhood of the origin.

Expression of the resonant term

The voice is driven from the bottom bar.

The equation obtained after the three changes of variables is ; (pay attention to the fact that we are working in $\mathbb{C}$!) :

$$\boxed{\dot{\xi} = \lambda\xi + h\xi^2\overline{\xi} + O_4(|\xi|^4).}$$

Setting : $\xi = re^{i\varphi}$ et $h = h_r + ih_i$:

$$\boxed{\dot{r} = ar + h_r r^3 \text{ et } \dot{\varphi} = b + h_i r^2.}$$

A calculus leads to the solution (neglecting terms of order 4) : Calculus

$$\bullet \text{ Si } h_r < 0 : \begin{cases} \lim_{t\to\infty} = r_{lim} = \sqrt{\frac{a}{-h_r}} \\ \lim_{t\to\infty} \dot{\varphi} = b + h_i r_{lim}^2 \end{cases} \quad \bullet \text{ si } h_r \geq 0 : \quad \lim_{t\to\infty} r(t) = \infty,$$

$$r(t) = \frac{ar_0 e^{at/2}}{\sqrt{a + h_r r_0^2(1 - e^{at})}}, \quad \varphi = \varphi_0 + bt - \frac{ah_i}{h_r}\mathrm{Log}\left(1 + \frac{h_r r_0^2}{a}(1 - e^{at})\right).$$

$\dot{\varphi}_{lim}$ is the phase velocity and r_{lim} is the radius of the limit cycle.

Signature of the resonant term

The voice is driven from the bottom bar.

Remark (How to make use of the previous result)

In case of a limit cycle ($h_r < 0$) it is obtained as an asymptotic curve excepted if the initial condition is on this cycle. One gets the convergence of the trajectory for the full non linear model to the limit cycle as soon as the initial condition is in a neighbourhood of the origin.
At the opposite ($h_r > 0$) its is necesary to go further in the algorithm untill one finds a resonant term which can (may be) stabilize the solution.

Remark (Algorithm)

Due to the enormous complexity of the computations in the changes of variables, it is highly recommended to use symbolic computation softwares. It is worth to notice that in the example considered there is no resonant term at order 4 but only at the order 5.

Signature of the resonant term

The voice is driven from the bottom bar.

The solution found for $h_r < 0$ can be written :

$$\xi_{lim}(t) = r_{lim} e^{i\dot{\varphi}_{lim} t}.$$

If one uses a Fourier transform there is a resonance for $\omega_r = \dot{\varphi}_{lim}$. Due to the third change of variables which is : $z = \xi + q(\xi, \overline{\xi})$ one can claim that the solution in z will contain resonances for :

$$\omega \in \{\omega_r, 3\omega_r\}.$$

Then from $y = z + p(z, \overline{z})$ one deduces that the spectrum of y contains resonances for :

$$\omega \in \{\omega_r, 2\omega_r, 3\omega_r, 4\omega_r, 6\omega_r\}.$$

A nice method for counting the harmonics is to use a Nyquist diagram in time. An example

The frequency gap observed for $5\omega_r$ is a caracteristic of the resonant term of order 3. This will be discussed in the example of a military aircraft in the following.

With symbolic computation...

The voice is driven from the bottom bar.

Let us consider the simplified case :

$$f(x,\dot{x}) = A + B\dot{x} + Cx + Dx^2 + Ex\dot{x} + F\dot{x}^2 + Gx^3 + Hx^2\dot{x} + Ix\dot{x}^2 + J\dot{x}^3.$$

The first change of variables (linear) enables one to obtain an equation as : ($\frac{a}{2}$ is assumed to be small compared to $\sqrt{\omega^2 - C}$) :

$$\boxed{\begin{array}{l} \dot{y} = \lambda y + g(y,\overline{y}), \text{ with } \lambda = a + ib = a \pm i\sqrt{(\omega^2 - C)^2 - \frac{a^2}{4}} \\ \text{solution of } \lambda^2 - B\lambda + (\omega^2 - C) = 0 \\ \text{and } y = \frac{1}{2}(x + \frac{i}{b}(ax - \dot{x})) \text{ ou } x = y + \overline{y},\ \dot{x} = \lambda y - \overline{\lambda y}. \end{array}}$$

We apply the algorithm to :

$$\dot{y} = \lambda y + g(y,\overline{y}),$$

and we denote by L_2 (respectivelyL_3) the matrix which appears in the two non linear changes of variables.

Organisation of the program

The voice is driven from the bottom bar.

An algorithm for the computation of terms of order 3 in the normal form method : A useful reference...

$$\boxed{g(y,\overline{y}) = Dy^2 + Ey\overline{y} + F\overline{y}^2 + Gy^3 + Hy^2\overline{y} + Iy\,\overline{y}^2 + J\,\overline{y}^3 + \dots}$$

$\downarrow$

Computation of p as a function of fonction D,E,F

$\downarrow$

Computation of the terms of order 3

$\downarrow$

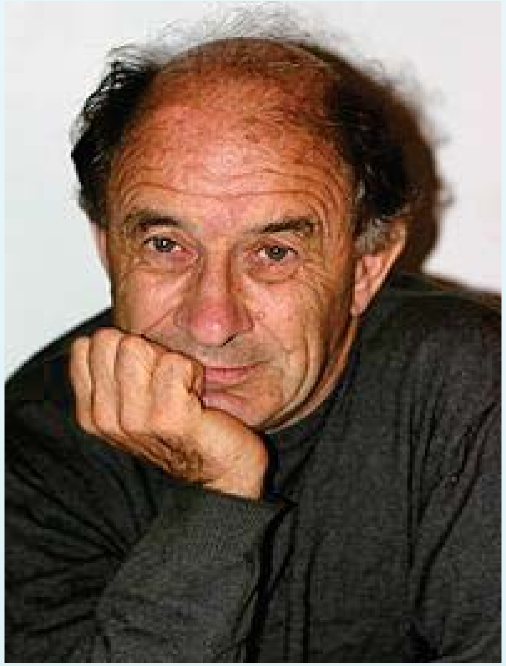
Computation of the coefficient of $\xi^2\overline{\xi}$

$\downarrow$

Expression of the limit cycle of oscillation in ξ and then in x

Questionnaire d'assimilation

The voice is driven from the bottom bar.



Vladimir Arnold one of the pilar of dynamical systems.

Answer to the questions and check your score.

Run the qcm

Evitez de regarder les réponses trop vite !

An example

The voice is driven from the bottom bar.

Lecture 4: Study of a reduced model in a wind tunnel

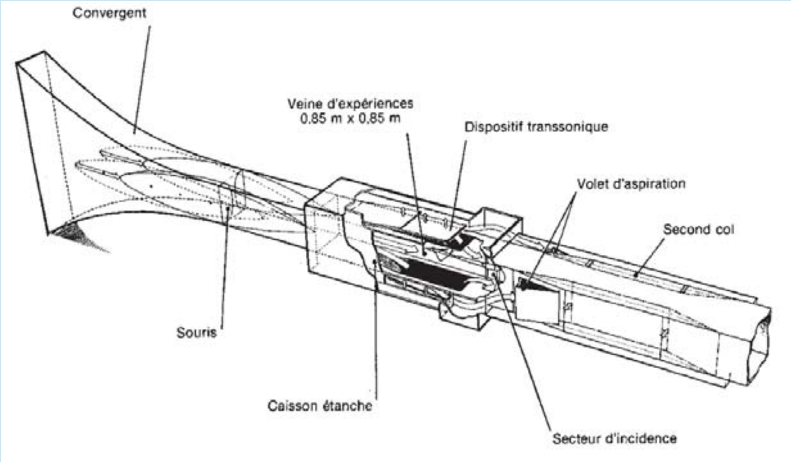
PLAN of lecture 4

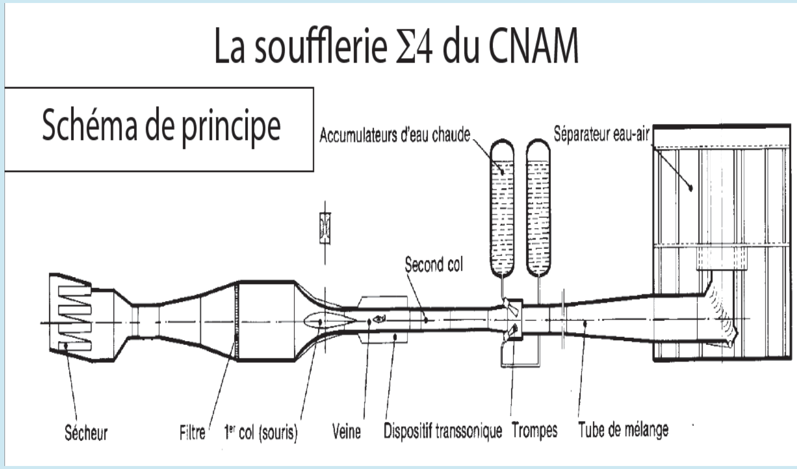
Energetical principle

The voice is driven from the bottom bar.

The wind tunnel used

Zoom1 ↓ Zoom2 ↓ ↓





Caractéristiques de la soufflerie

Type de soufflerie	à rafales à circuit ouvert
Sections de la veine d'essai	0,90 m x 0,85 m
Vitesses du vent	0,3 < Mach < 2,8
Réglage de la vitesse du vent	2 cols soniques : - souris à l'amont, parois vertical à l'aval - variation continue en monté comme en descente
Parois	perméables ou pleines selon le nombre de Mach
Système énergétique	aspiration à l'aval par trompe à eau chaude
Pression génératrice	1 bar
Durée moyenne d'une rafale	60 secondes
Nombre de Reynolds	1,5 10^6 à Mach = 1,2 (d = 0,10 m)

Zoom ↑

The tests were performed in 1990 by the Scientific computation team at CNAM and financed by the DGA

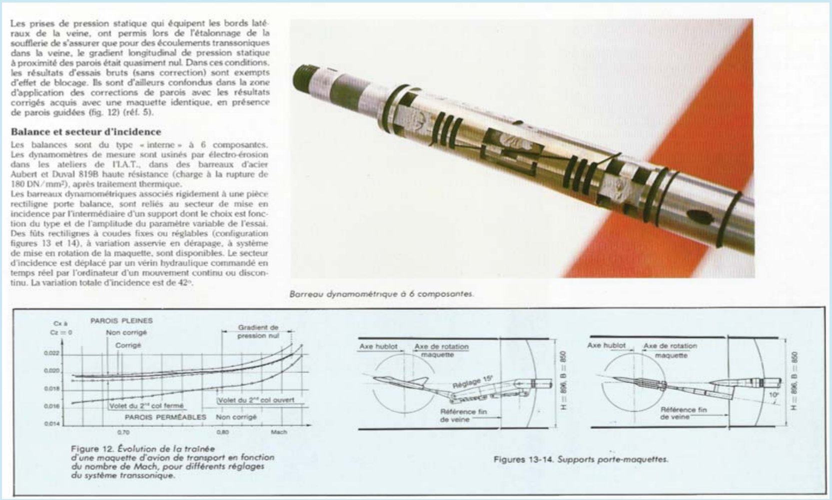
Ph. D. and M. T. Ribereau [1996], Non linear dynamics of test models in wind tunnels, in Eur. J. Mech. A/Solids,15, n$^{\circ}$ 1, p. 91-136.

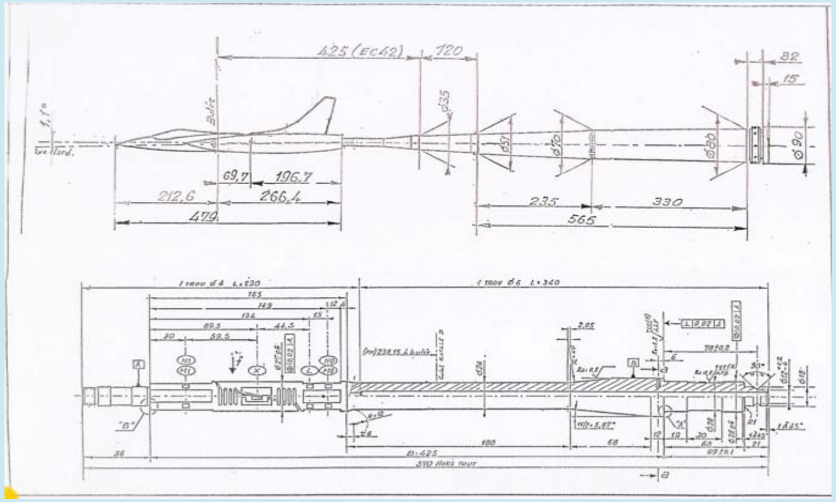
The tests

The voice is driven from the bottom bar.

Supporting system

Zoom1 Zoom2





Measures and their treatment

The two gadge bridges are equiped by electric resistors and set in Wheaston bridges which enable to measure the six components of the forces.

The frequency used in the sampling is larger than 256 Hz and a Hanning filtering is applied. Hence the results are reliable up to 130 Hz

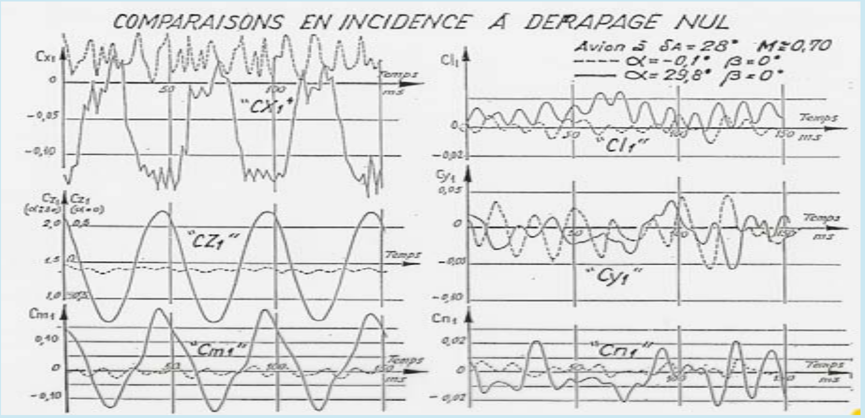
The curves obtained are smoothed using polynomial of degree 120.

The measures

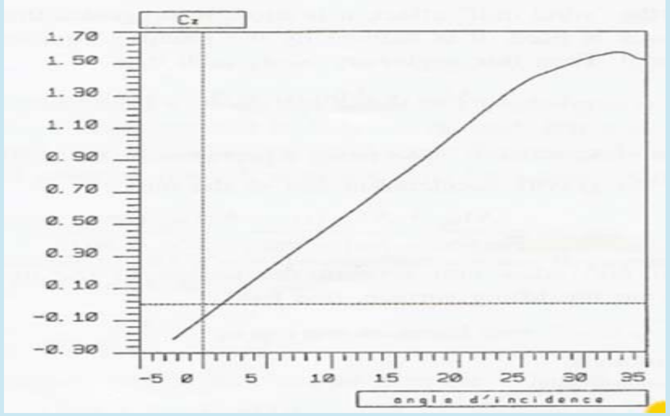
The voice is driven from the bottom bar.

Lecture 4: Study of a reduced model in a wind tunnel

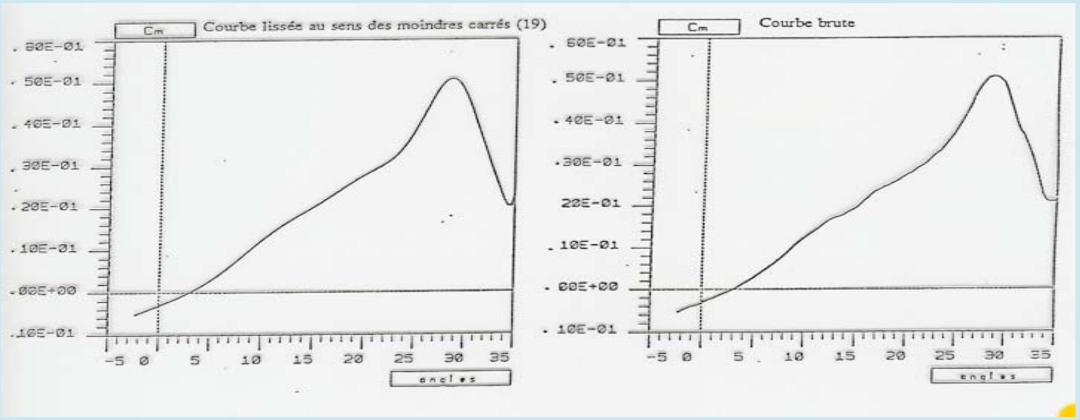
Mesures de $c_x, c_z, cm \ldots$ Zoom

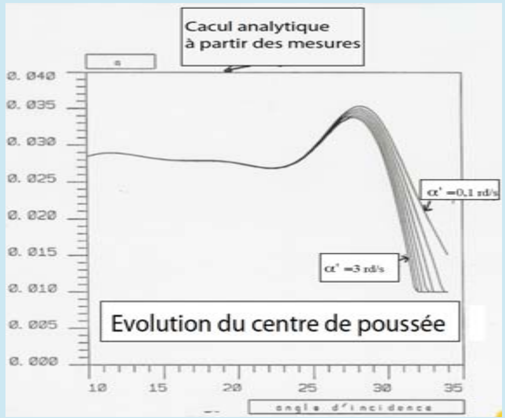


Measure of c_z Zoom



Pitching coefficient cm_0 and position of the aerodynamical center. Zoom1 Zoom2





What has been seen

The voice is driven from the bottom bar.

The aerodynamical center is perfectly defined in 2D. It is the point where the resultant aerodynamical forces are applied. Its distance from O is a.It is defined by:

$$\boxed{ac_z(\alpha) + Lc_{m0}(\alpha) = 0}$$

The smoothing is necessary because it is required to take third order derivatives of the aerodynamical coefficients. In fact only for c_{m0} in our case.

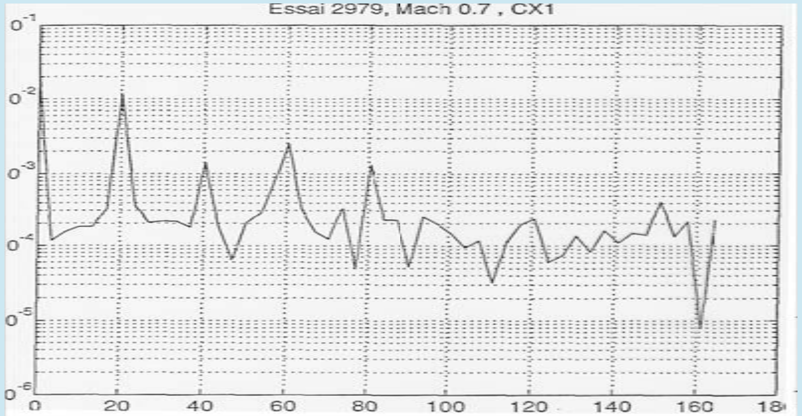
First conclusions

The measures reported on the previous screen shows that the pitching moment increases a lot before an angle of attack close to 28^0. The heaving movement seems to be a single frequency it is not the case for the pitching angle. This proves clearly that the movement can't be modelled by a linear system with one degree of freedom.

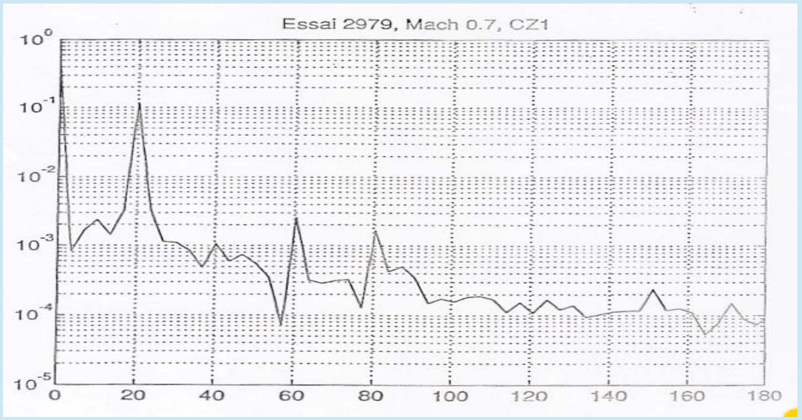
FFT of the measures

The voice is driven from the bottom bar.

FFT de c_x Zoom



FFT de c_z Zoom



Comments

We observe that the energy is discharged on the fundamental eigenmode of the pendulum system (20 Hz) but also on the harmonics $2, 3, 4$ *et* 6. *It is worth to notice that there is no energy on the fifth harmonic. Hence the analysis given in the previous lecture enables one to confirm the existence of a resonant term of order 3 and the explanation of the limit cycle given here using the the apparent wind velocity.*

Notations:

- J_0 Inertia around point o;
- M Mass of the system;
- C Bending stiffness of the support;
- a Distance between o and the aerodynamic center.
- α_a Apparent pitching angle;
- V_a Apparent velocity;
- ϱSL Mass density times the reference volume.

A simple non linear model with 1 DOF

$$J_0\ddot{\alpha} + C(\alpha - \alpha_0) = \frac{\varrho SL|V_a|^2}{2} c_{m0}(\alpha_a)$$

$$\alpha(0) = \alpha_0, \ \dot{\alpha}(0) = \alpha_1.$$

$$\frac{\varrho SL|V_a|^2}{2} c_{m0}(\alpha_a) = \frac{\varrho SL|V|^2}{2} c_{m0}(\alpha_0) - \frac{a\varrho SLV}{2}$$

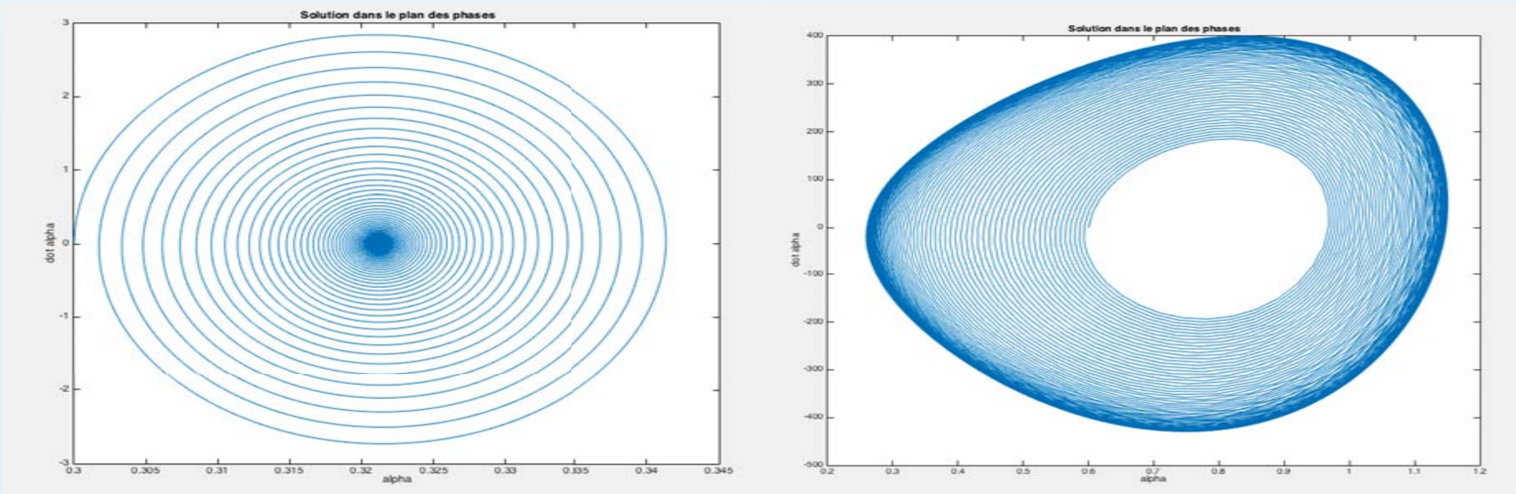
$$(2\sin(\alpha_0) c_{m0}(\alpha_0) + \frac{\partial c_{m0}}{\partial \alpha}(\alpha_0)\cos(\alpha_0))\dot{\alpha} + \dots$$

Direct simulation of the 1 DOF model

The voice is driven from the bottom bar.

- A simple *Matlab* code has been used with a time step integration scheme. The scheme is: Simul. (In this presentation we didn't use the exact data from the military aircraft)

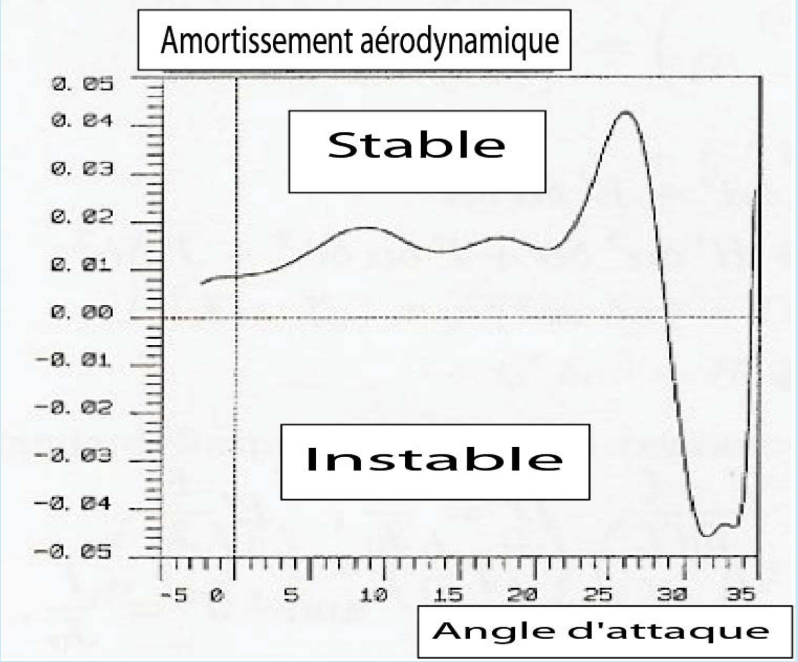
$$\boxed{\frac{\alpha^{n+1} - 2\alpha^n + \alpha^{n-1}}{\Delta t^2} + c(\alpha^n - \alpha_0) = \frac{\varrho SL|V_a|^2}{2} c_{m0}(\alpha_a).}$$



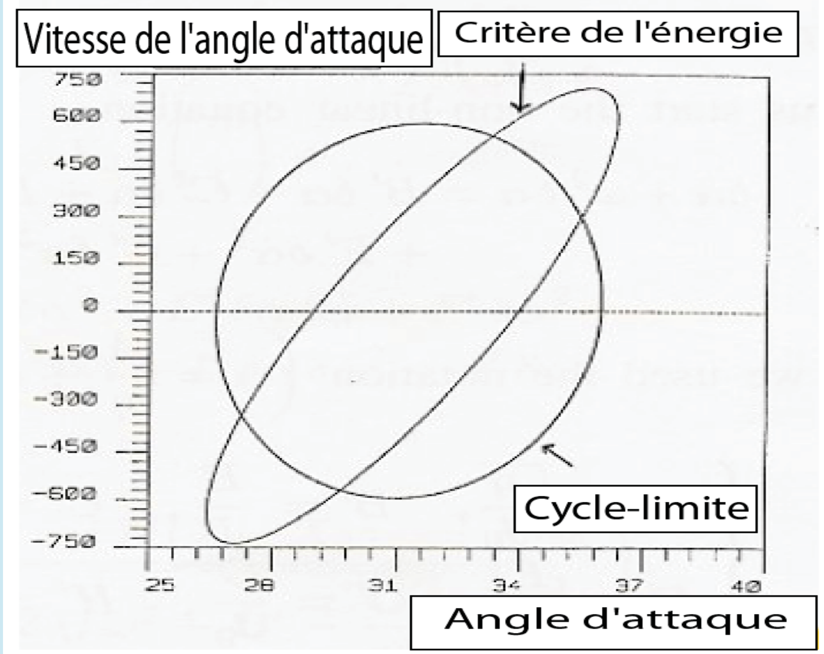
- In a second step we have built the normal form of the model using *Mathematica* and around 28^o.
  We obtained a resonant term of order 3.

The voice is driven from the bottom bar.

Aérodynamical damping.



Limit cycle and the energy criterion



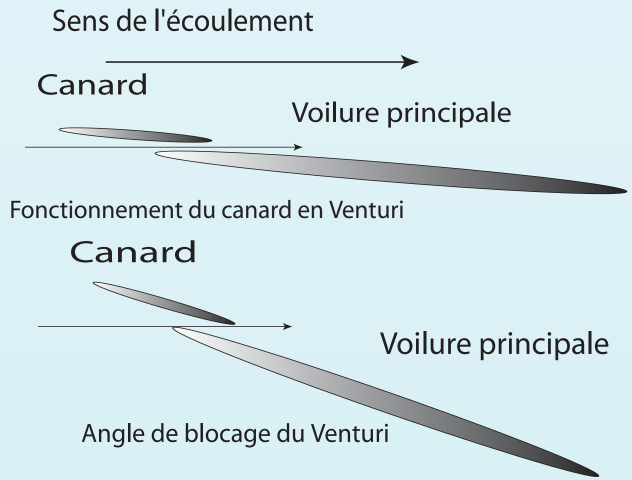
The negative damping appears for angle of attack α between 28^o and 34^o. This is exactly what has been observed in the wind tunnel tests. Furthermore the shape and amplitude of the limit cycle is the one observed. Finally the FFT of the measures shows the gap of energy on the fifth harmonic (see Lecture 3).

Conclusion

The voice is driven from the bottom bar.

A simple one DOF model enables to reproduce the phenomenon observed in the wind tunnel with a surprising accuracy. In fact we performed three tests: the two first in 1989 at $M = .7$ led to the breakdown of the reduced model after 60 sec. The third one was performed in 1991 at a reduced velocity ($M = .5$) and stopped as soon as the limit cycle was observed (30 sec.).

But a remaining question was the understanding of the sudden decrease of the pitching coefficient involving the stall flutter phenomenon which is not really seen on the lift coefficient. In fact it appeared that the uncontrolled *Canard* works as Venturi (jib in sailing). It improves the local lift on the front of the main (delta) wing and thus the pitching moment. But at 28^o the Venturi is suddenly closed and the pitching moment decreases. Hence the plane moves down, the Venturi opens again and so on inducing the limit cycle of oscillation and the breakdown of the reduced model.



QCM

The voice is driven from the bottom bar.

Reduced model of a military aircraft in our wind tunnel ($\leq 65m/s$) S10 du CNAM.

Answer to the questions and check your score.

Run the qcm

Do not look at the answer too quickly!