Delay control of reaction-diffusion equations

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Keywords: optimal control, parabolic equation, optimal delay, nonlocal time delay, measure control

Mathematics Subject Classification (2010): 49K20, 35K58, 49M25

In 1992, Pyragas [6] introduces a simple concept of feedback control to stabilize the behaviour of solutions of dynamic systems of the form $y'(t) = F(y)$ that exhibit chaotic patterns. He shows that the perturbed system

$$y'(t) = F(y) + \kappa(y(t-s) - y(t))$$

can be stabilized for some weight κ and some delay s chosen heuristically.

It is also of interest in applications –e.g., Lasers [7], chemical reactions [3] or Parkinson's desease [5]– the delay control of reaction-diffusion systems. A general way to do this is to look for a control $u \in \mathcal{M}[0,T]$ such that the solution of the equation

$$\begin{aligned} \partial_t y - \Delta y + R(y) &= \int_0^T y(x,t-s)du(s) \text{ in } \Omega \times (0,T) \\ \partial_n y &= 0 \text{ on } \partial\Omega \times (0,T) \\ y(x,t) &= y_0(x,t) \text{ for a.e. } x \in \Omega \text{ if } t \le 0 \end{aligned}$$

minimizes $J(u) = 0.5\|y_u(x,t) - y_d(x,t))\|^2_{L^2(Q)} + \alpha\|u\|_{\mathcal{M}[0,T]}$ for some target state y_d and some parameter $\alpha > 0$, being $Q = \Omega \times (0,T)$. This formulation, which is studied in [1], has the advantage of being very general. Both classical Pyragas delay controls, taking $u = \kappa(\delta_s - \delta_0)$, and nonlocal Pyragas delay controls of the form $\kappa(\int_0^T \int_\Omega g(x,s)y(x,t-s)ds - y(x,t))$, studied in [4], are included in this formulation.

Numerical experiments show that the solution of such a measure control approach is usually a combination of several Pyragas controls. One of the disadvantages of the previous method is that, in principle, the number of delays cannot be imposed beforehand. In practice, the number of delays can be limited. From the numerical point of view, the appearance of the non-differentiable $\|u\|_{\mathcal{M}[0,T]}$ term leads to some difficulties. Also, the discretization of the measure makes that the delays found numerically must be located at the time mesh nodes.

A more practical, though less general approach is to fix $m \in \mathbb{N}$ and look for delays and weights $(s_i, \kappa_i)_{i=1}^m$ such that the solution of the equation

$$\begin{aligned} \partial_t y - \Delta y + R(y) &= \sum_{i=1}^m \kappa_i (y(x,t-s_i) - y(x,t)) \text{ in } Q \\ \partial_n y &= 0 \text{ on } \partial\Omega \times (0,T) \\ y(x,t) &= y_0(x,t) \text{ for a.e. } x \in \Omega \text{ if } t \le 0 \end{aligned}$$

minimizes $J(s,\kappa) = 0.5\|y_{s,\kappa} - y_d\|^2_{L^2(Q)}$. Despite looking like a particular case of the previous problem, the control, that before appeared as a bilinear control, now appears in a highly nonlinear way. Differentiability properties cannot be deduced from the general case.

For this problem, we can prove, see [2], continuity of the control-to-state mapping and also compute its partial derivatives. Using these and an appropriate adjoint state equation, we can compute the partial derivatives of the objective functional. Moreover, we show that with an adequate discretization, using a standard Galerkin method in space and a Petrov-Galerkin scheme in time –continuous elements for the state and discontinuous for the adjoint state–, the approaches optimize-then-discretize and discretize-then-optimize coincide. In this way, we compute the partial derivatives of the discrete functional in an effective way.

Finally, some numerical examples illustrate the theoretical results and the applications of this technique.

Acknowledgements

The first two authors were partially supported by the Spanish Ministerio de Economía y Competitividad under research projects MTM2014-57531-P and MTM2017-83185-P. The third author was supported by the collaborative research center SFB 910, TU Berlin, project B6.

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