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Review of numerical schemes for the classical Keller-Segel system

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The movement of biological cells in response to chemical signals, also called as chemotaxis, was modelled by Keller-Segel in 1970. Although there are several models, we will focus on the classical one, which is given by the following equations:

$$(1)\quad \begin{cases} u_t = \Delta u - \alpha_1 \nabla \cdot (u \nabla v), & x \in \Omega,\ t > 0, \\ v_t = \alpha_2 \Delta v - \alpha_3 v + \alpha_4 u, & x \in \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), v(x,0) = v_0(x), & x \in \Omega, \end{cases}$$

being u and v two non-negative functions which represent the density of cells and chemical-signal, respectively.

Recently, a lot of research in this subject has been done from an analytical point of view (see e.g. [1] and references therein). Global in time existence and boundedness of the solution has been show if the initial data is small enough, while blow-up in some solutions of (1) occurs in many other interesting cases.

In this work we focus on numerical schemes for the Keller-Segel model. We review some time schemes from the point of view of energy-stability and positivity and we test them under different finite element space discretizations, including Lagrange, Bernstein and high order Lobatto hierarchical familes.

Let us introduce a semi-discretization in time, considering a partition of the time interval $(0, T)$ into subintervals of size $k > 0$. In each time step t^{m+1}, we approximate u_t and v_t by implicit Euler schemes as follows:

$$(2)\quad \begin{cases} u^{m+1} - k\Delta u^{m+1} + k\nabla \cdot (u^{m+r_1} \nabla v^{m+r_2}) = u^m, \\ v^{m+1} - k\Delta v^{m+1} + k\nabla v^{m+r_3} + k u^{m+r_4} = v^m, \end{cases}$$

where $r_1, r_2, r_3, r_4 \in \{0, 1\}$ and we fix $\alpha_i = 1$ $(i = 1, \dots, 4)$.

Here we analyze those tuplas (r_1, r_2, r_3, r_4) that result in linear uncoupled energy-stable schemes. Specifically, let us consider

$$E_m = \int_\Omega \left(u^m \log(u^m) - u^m v^m + \frac{1}{2}(v^m)^2 + \frac{1}{2}(\nabla v^m)^2 \right),$$

$$D_m = \int_\Omega \left| \frac{\nabla u^m}{\sqrt{u^m}} - \sqrt{u^m} v^m \right|^2 + \int_\Omega (\delta_t v^m)^2,$$

$$N_m = \frac{k}{2} \int_\Omega \left((\delta_t (\nabla v^m))^2 + (\delta_t v^m)^2 \right)$$

for any $M \in \mathbb{N}$. Then one can assert the following result:

THEOREM 1 *Let Sch_1 and Sch_2 be the respective uncoupled schemes related to $(r_1, r_2, r_3, r_4) = (1, 0, 1, 0)$ and $(r_1, r_2, r_3, r_4) = (1, 1, 1, 0)$. The following energy law holds:*

$$\frac{d}{dt} E_{m+1} + M_{m+1} \leq -D_{m+1} - N_{m+1},$$

being

$$M_{m+1} = \frac{k}{2} \int_\Omega \left((u^{m+1})^2 + (\nabla v^{m+1} \delta_t v^{m+1})^2 \right)$$

and $M_{m+1} = 0$ for Sch_1 and Sch_2 respectively. In addition, one has positivity of the solutions of Sch_1 and Sch_2.

For the space discretization, we made computational experiments for Sch_1, Sch_2 and other linear schemes resulting from (2). Specifically, we implemented a systematic suite of tests utilizing the FreeFem++ language [2, 3] for Lagrange Finite Elements up to order $p = 4$ and the C++ library Libmesh [4] for (positive) Bernstein and (hierarchical) Lobatto elements up to order $p = 6$ and $p = 10$, respectively. Experiments included 2D and 3D cases and different indicators were studied, including tests of energy, positivity and blow-up. These experiments will be presented, obtaining some conclusions.

References

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