An inverse problem governed by the Lamè System: tumor identification

Enrique Fernández-Cara* and Faustino Maestre*

Keywords: Inverse problems; linear elasticity; Lamè systems; bounded variation coefficients; elastography

Mathematics Subject Classification (2010): 35L55; 35R30; 49N45; 74B05

Considering that the stiffness of the health or illness tissues is different, it could help us to detect the presence of disease. One important motivation for the medical imaging techniques comes from cancer detection: roughly speaking, a tumor tissue is 5 to 28 times stiffer than normal soft tissue and, consequently, the resulting deformation after a mechanical action is smaller.

The medical imaging techniques are a non-invasive techniques and can be used to investigate diseases in internal organs as well as, for instance, in the detection and diagnosis of liver, thyroid and prostate cancers. In particular, elastography is a medical imaging technique which gives information about the elastic properties of the tissue

From the mathematical point of view, the relevant questions can be formulated in terms of an inverse problem governed by a system of PDEs (see [1],[2],[3]). In our work, we fix an open, bounded and smooth domain $\Omega \subset \mathbb{R}^N$, with $N = 2$ or $N = 3$ and a positive time T. We assume that f is an external source acting on the body, (u^0, u^1) is the initial displacement-velocity pair of the particles, B is the dispalcement field the boundary and the system es governed by the *Lamè system*:

$$(1)\begin{cases} u_{tt} - \nabla \cdot \big(\mu(x)(\nabla u + \nabla u^T) + \lambda(x)(\nabla \cdot u)\mathbf{Id.}\big) = f, \\ \hfill (x,t) \in \Omega \times (0,T) \\ u = 0, \hfill (x,t) \in \Sigma \\ u(x,0) = u^0(x),\ u_t(x,0) = u^1(x), \hfill x \in \Omega. \end{cases}$$

The inverse problem consists of finding the coefficients $\lambda = \lambda(x)$ and $\mu = \mu(x)$, such that the solution to (1) satisfies the additional condition

$$(2)\ (\mu(x)(\nabla u + \nabla u^T) + \lambda(x)(\nabla \cdot u)\mathbf{Id.}) \cdot \nu = \Upsilon \text{ on } S \times (0,T).$$

where $\nu = \nu(x)$ is the outwards directed unit normal vector at points $x \in \partial\Omega$, $S \subset \partial\Omega$ and Υ is prescribed.

The analysis and search of a solution to an inverse problem is usually hard and, sometimes, impossible. In this work, we propose a reconstruction method relying on the analysis of a *direct problem* related to (1) and (2).

Thus, let us introduce the cost function

$$I(\lambda,\mu) = \frac{1}{2}\int_0^T \| \big(\mu\nabla u + \nabla u^T) + \lambda(\nabla \cdot u)\mathbf{Id.}\big) \cdot \nu - \Upsilon \|^2 \, dt,$$

where $\| \cdot \|$ is the norm in $H^{-1/2}(S)^N$ and let us consider the following direct problem

$$(3)\qquad \begin{cases} \text{Minimize } I(\lambda,\mu) \\ \text{Subject to (1).} \end{cases}$$

It is clear that $I(\lambda^\star, \mu^\star) = 0$ if and only if $(\lambda^\star, \mu^\star)$ solves the inverse problem (1)–(2).

In this work, we prove that optimization problem (3) admits optimal solutions under the assumptions of uniform bound for the total variation of the admissible pairs (λ, μ). Finally we show some numerical experiments where the minimization problem (3) is solved for different data and recovring the numerical evidence that cost function I vanishes at the minimizer.

Acknowledgements

This research was partially supported by Ministerio de Economía y Competitividad under grant MTM2016-76990-P and MTM2014-53309-P with the participation of FEDER.

References

[1] Ammari H. et al., Reconstruction of small interface changes of an inclusion from modal measurements II: The elastic case, J. Math. Pures Appl. (9) 94 (2010), no. 3, 322 - 339.

[2] Imanuvilov O. Y. and Yamamoto M., On reconstruction of Lamè coefficients from partial Cauchy data, J. Inverse Ill-Posed Probl. 19 (2011), no. 6, 881- 891.

[3] Fernández-Cara, E. and Maestre F. An inverse problem in elastography involving Lamè systems. Journal of Inverse and Ill-posed Problems, Published on-line. doi:10.1515/jiip-2017-0065

*Departamento de Ecuaciones Diferenciales y Análisis Numérico, Universidad de Sevilla, Avda. Reina Mercedes s/n, 41080 Sevllla (SPAIN). Email: cara@us.es, fmaestre@us.es