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Delay control of reaction-diffusion equations-12

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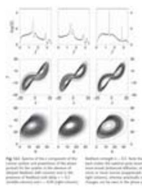
Delay reaction-diffusion equations with kernel $u \in \mathcal{M}[0,T]$

$$\begin{cases} \partial_t y - \Delta y + R(y) = \int_{[0,T]} y(x,t-s)\,\mathrm{d}u(s) & \text{in } Q = \Omega \times (0,T), \\ \partial_n y = 0 & \text{on } \Sigma = \Gamma \times (0,T), \\ y(x,t) = y_0(x,t) & \text{in } Q^- = \Omega \times [-T,0], \end{cases}$$

- Pyragas feedback control stabilization: $\sum \kappa_i (y(x,t-s_i) - y(x,t))$
- Nonlocal control: $\int_0^T y(x,t-s) g(s) ds$
- Applications: Laser control, Chemical waves, Parkinson disease

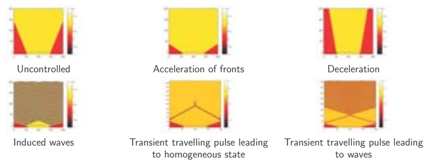
Physics literature: Tune parameters and solve direct problem

Pyragas control of chaos



Reproduced from *Handbook of Chaos Control* E Schöll and H.G. Schuster eds.

Chemical wave (Schögl model)



Data from Löber, Coles, Siebert, Engel, Schöll, arXiV:1403.3363v1. (Pictures: self-elaboration)

Analysis of the problem: State equation

- $\forall u \in \mathcal{M}[0,T] \Rightarrow \exists$ unique $y_u \in Y = L^2(0,T;H^1(\Omega)) \cap C(\bar{Q})$
- If $y_0(\cdot,0) \in H^1(\Omega)$ and Ω is convex then $y_u \in H^{2,1}(Q)$
- If $u_k \stackrel{*}{\rightharpoonup} \bar{u}$ in $\mathcal{M}[0,T]$; then $y_k \to \bar{y}$ in Y. ($\Rightarrow$ existence of optimal solution)

MEASURE CONTROL

- $u \in \mathcal{M}[0,T] \mapsto y_u \in Y$ is C^1
- Apply implicit function theorem.
- Proof technique similar to other control problems.

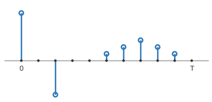
OPTIMAL DELAY

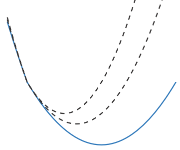
- $(s,\kappa) \in \mathcal{C} \mapsto y_{(s,\kappa)} \in Y$ is C^0
- We have partial derivatives.
  Difficult pass to the limit: use results about average continuity of Nečas 1967.

Discretization issues

MEASURE CONTROL

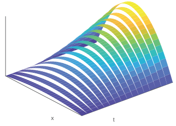
- Combination of Dirac measures centered on the time nodes to discretize the measure. (We prove weak* convergence in $\mathcal{M}[0,T]$, the best possible in this case!)





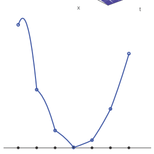
- Write the subgradient condition $\bar\lambda \in \partial j(\bar u)$ using a semismooth function and solve the optimality system using a semismooth Newton method.
- Use Tykhonov regularization and a continuation technique.

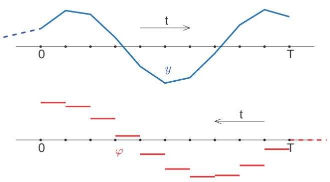
- Solve delay PDEs: discontinuous elements in time and continuous elements in space: dG(0)cG(1) discretization.



OPTIMAL DELAY

- dG(0) (time) leads to:
  - ▷ Discrete functional non-differentiable at every time node.
  - ▷ discretize-then-optimize $\neq$ optimize-then-discretize





- Instead, use Petrov-Galerkin in time
  - ▷ Continuous elements for the state
  - ▷ Discontinuous for adjoint state

Control problems: look for u to approach target y_d

$$F(y) = 0.5\|y - y_d\|^2_{L^2(Q)}$$

MEASURE CONTROL

$$J_1(u) = F(y_u) + \nu j(u)$$

- Bilinear control
- Acting as measure kernel
- **Non differentiable convex term** $j(u) = \|u\|_{\mathcal{M}[0,T]}$
- Promotes sparsity: usually the optimal $\bar u$ is a combination of Dirac measures

OPTIMAL DELAY

Fix $m \in \mathbb{N}$.

$\mathcal{C} = \{(s,\kappa) \in\subset \mathbb{R}^{2m} : 0 < s_i \leq b < T\}$

$$u(s,\kappa) = \sum_{i=1}^m \kappa_i(\delta_{s_i} - \delta_0)$$

- **Better not treat as subcase:**
  - ▷ U_{ad} would not be convex
  - ▷ $s \mapsto \delta_s$ not differentiable
- Change to finite-dimensional optimization

$$J_2(s,\kappa) = F(y_{u(s,\kappa)}) + \frac{\nu}{2}|\kappa|^2$$

- Both problems have at least a global solution.

Analysis of the control problem

Adjoint state equation

$$\begin{cases} -\partial_t \varphi - \Delta\varphi + R'(y_u)\varphi = \int_{[0,T]} \varphi(x,t+s)\,\mathrm{d}u(s) + y_u - y_d & \text{in } Q \\ \partial_n \varphi(x,t) = 0 & \text{on } \Sigma \\ \varphi(x,t) = 0 & \text{if } t \geq T \end{cases}$$

MEASURE CONTROL

$$\partial_u F(y_u)(s) = \int_Q \varphi(x,t) y(x,t-s)\,\mathrm{d}x\,\mathrm{d}t \quad \forall s \in [0,T]$$

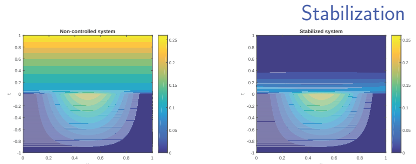
Optimality condition: $\exists$ a subgradient $\bar\lambda \in j(\bar u)$ such that $\partial_u F(y_{\bar u}) + \nu\bar\lambda = 0$

OPTIMAL DELAY

$\partial_{s_i} F(y_{u(s,\kappa)}) = -\kappa_i \int_Q \varphi(x,t)\partial_t y(x,t-s_i)dxdt$, $\partial_{\kappa_i} F(y_{u(s,\kappa)}) = +\kappa_i \int_Q \varphi(x,t)(y(x,t-s_i) - y(x,t))dxdt$

Examples: $R(y) = (y - y_{\text{metastable}})(y - y_{\text{unstable}})(y - y_{\text{stable}})$

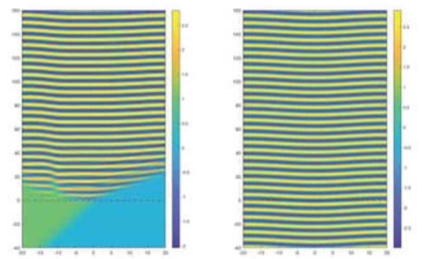
Stabilization



- Lead the system to an unstable equilibrium
- Optimal control of the form $\bar u = \kappa\delta_s$.

Leading a chemical wave to a prescribed target pattern

- Allow a time shift in the target
- Use $m = 5$ Pyragas controls.



Optimal state (left) and shifted target (right)

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