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Convergence rate of an asymptotic preserving scheme for the limit of the p-system with damping - N15

S.Bulteau[1], C.Berthon[1], M.Bessemoulin-Chatard[1]

[1]Laboratoire de Mathématiques Jean Leray, University of Nantes

Introduction

p-system :

$$\begin{cases} \partial_t\tau - \partial_x u = 0, \\ \partial_t u + \partial_x p(\tau) = -\sigma u. \end{cases} \qquad (H)$$

Unknowns
- Specific volume $\tau > 0$,
- Velocity $u \in \mathbb{R}$.

Data
- Pressure p,
- Friction σ.

Regime studied
- Long time,
- High friction.

$\Rightarrow$ Rescaling with a small ε : $t \leftarrow t/\varepsilon$, $\sigma \leftarrow \sigma/\varepsilon$, $u \leftarrow \varepsilon u$.

Rescaled system :

$$\begin{cases} \partial_t\tau - \partial_x u = 0, \\ \varepsilon^2\partial_t u + \partial_x p(\tau) = -\sigma u. \end{cases} \qquad (H_\varepsilon)$$

Porous media equation $\varepsilon \to 0$:

$$\begin{cases} \partial_t\tau - \partial_x u = 0, \\ \partial_x p(\tau) = -\sigma u. \end{cases} \qquad (P)$$

- Lattanzio and Tzavaras [1] : convergence from weak entropy solutions of (H_ε) to smooth solutions of (P) with a rate of order ε^2 for the $L^2_x(\mathbb{R})$ norm.
- Berthon, Bessemoulin and Mathis [2] : extension to the semi discrete framework.

Motivation : Obtain a **non trivial** discrete analogous result.

Numerical scheme

Jin *et al* [3] : reformulation of (H_ε)

$$\begin{cases} \partial_t\tau - \partial_x u = 0, \\ \partial_t u + \partial_x p(\tau) = -\frac{1}{\varepsilon^2}\left(\sigma u + (1-\varepsilon^2)\,\partial_x p(\tau)\right). \end{cases}$$

Two time splitting method :
- Approximation of the solutions of a conservative hyperbolic problem independant of ε.
- Time implicit approximation of the source term.

Scheme for (H_ε) :

$$\begin{aligned} \tau_i^{n+1} &= \tau_i^n + \Delta t\delta_x u_i^n + \frac{\lambda}{2}\Delta x\Delta t\delta_{xx}\tau_i^n, \\ \varepsilon^2 u_i^{n+1} &= \varepsilon^2 u_i^n - \sigma\Delta t u_i^{n+1} - \Delta t\delta_x p(\tau^{n+1})_i \\ &\quad + \varepsilon^2\Delta t^2\delta_{tx}p(\tau_i^{n+\frac12}) + \frac{\lambda}{2}\varepsilon^2\Delta x\Delta t\delta_{xx}u_i^n. \end{aligned} \qquad (H_\varepsilon^\Delta)$$

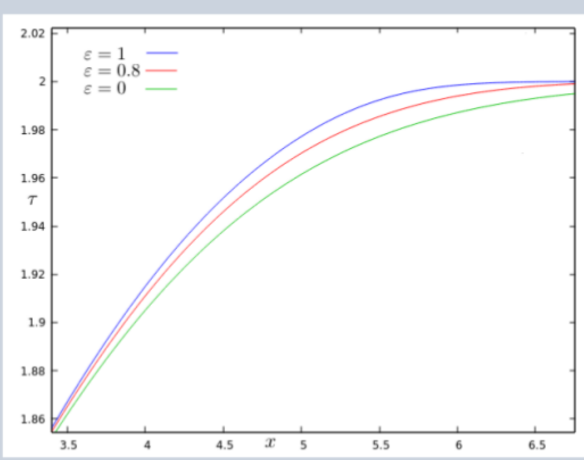
Limit scheme for (P) :

$$\begin{aligned} \overline{\tau}_i^{n+1} &= \overline{\tau}_i^n + \Delta t\delta_x\overline{u}_i^n + \frac{\lambda}{2}\Delta x\Delta t\delta_{xx}\overline{\tau}_i^n, \\ \delta_x p(\overline{\tau}^{n+1})_i &= -\sigma\overline{u}_i^{n+1}. \end{aligned} \qquad (P^\Delta)$$

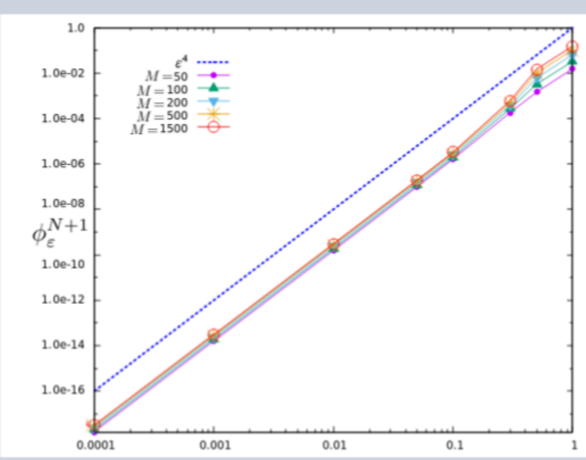
Numerical illustration

- $\sigma = 1$,
- $p(\tau) = \tau^{-\gamma}$ with $\gamma = 1, 4$,
- $T = 0.5$

- $\tau_0(x) = \begin{cases} 1 & \text{if } x \le 0, \\ 2 & \text{else.} \end{cases}$
- Neumann boundary conditions.



(a) Specific volume with respect to space



(b) Error with respect to ε in log scale

Relative entropy

Entropy-entropy flux pair :

$$\eta_\varepsilon(w) = \frac{\varepsilon^2}{2}u^2 - P(\tau), \qquad \psi(w) = up(\tau).$$

where P is a primitive of p.
We set w_ε a weak entropy solution of (H_ε).

Entropy inequality :

$$\partial_t\eta_\varepsilon(w_\varepsilon) + \partial_x\psi(w_\varepsilon) \le -\sigma u_\varepsilon^2$$

We set $\overline{w}$ a smooth solution of (P).

Relative entropy : quadratic term of the Taylor expansion of η_ε in $\overline{w}$

$$\eta_\varepsilon(w|\overline{w}) = \eta_\varepsilon(w) - \eta_\varepsilon(\overline{w}) - \nabla\eta_\varepsilon(\overline{w}).(w-\overline{w})$$

$$\Rightarrow \eta_\varepsilon(w|\overline{w}) = \frac{\varepsilon^2}{2}(u-\overline{u})^2 - P(\tau|\overline{\tau}),$$

where

$$\begin{aligned} P(\tau|\overline{\tau}) &= P(\tau) - P(\overline{\tau}) - p(\overline{\tau})(\tau-\overline{\tau}), \\ &= (\tau-\overline{\tau})^2 p'(c) \le 0 \end{aligned}$$

Error estimator :

$$\phi_\varepsilon(t) = \int_{\mathbb{R}}\eta_\varepsilon(w_\varepsilon|\overline{w})dx \approx \|w_\varepsilon - \overline{w}\|^2_{L^2_x}$$

Relative entropy flux :

$$\psi(\tau, u|\overline{\tau}, \overline{u}) = (u-\overline{u})\left(p(\tau) - p(\overline{\tau})\right).$$

Main result

Discrete quantities : $\eta_i^{\varepsilon,n} = \eta_\varepsilon(w_i^n|\overline{w}_i^n)$, $\phi_\varepsilon^n = \sum_{i\in\mathbb{Z}}\eta_i^{\varepsilon,n}\Delta x$.

Hypotheses :
- Non explosion or vaccum for specific volumes
- Regularity on $\overline{w}_i^n$ (H)
- Regularity on the pressure
- Parabolic CFL
- Non restrictive smallness on Δt and ε

$$\Rightarrow \phi_\varepsilon^{N+1} \le B\left(\phi_\varepsilon^0 + \|u^0 - \overline{u}^0\|^2_{L^2_x} + \varepsilon^4\right)$$

Sketch of the proof - Step 1

Notation : $\Delta A = A - \overline{A}$.

Write a discrete evolution law on $\eta_i^{\varepsilon,n}$:

$$\begin{aligned} \delta_t\eta_i^{\varepsilon,n+\frac12} + \frac{1}{\Delta x}\left(\psi^{n+1}_{i+\frac12} - \psi^{n+1}_{i-\frac12}\right) &= -\sigma\left(\Delta u_i^{n+1}\right)^2 \\ &\quad - \frac{\varepsilon^2}{2}\Delta t\left(\delta_t\Delta u_i^{n+\frac12}\right)^2 + R_i^n + S_i^n, \end{aligned}$$

$$\begin{aligned} R_i^n = &-\varepsilon^2\Delta u_i^{n+1}\delta_t\overline{u}_i^{n+\frac12} + \varepsilon^2\Delta t\Delta u_i^{n+1}\delta_{tx}p(\overline{\tau})_i^{n+\frac12} \\ &+ \frac{\lambda}{2}\varepsilon^2\Delta x\Delta u_i^{n+1}\delta_{xx}\overline{u}_i^n, \end{aligned}$$

$S_i^n = \sum_{p=1}^{10} T^n_{p,i}$ is composed of ten terms to estimate.

Sketch of the proof - Step 2

Discrete integration in time and space :

$$\begin{aligned} \phi_\varepsilon^{N+1} - \phi_\varepsilon^0 \le &-\sigma\sum_{n=0}^{N}\|\Delta u^{n+1}\|^2_{L^2_x}\Delta t \\ &+ \sum_{n=0}^{N}\sum_{i\in\mathbb{Z}}R_i^n\Delta x\Delta t + \sum_{n=0}^{N}\sum_{i\in\mathbb{Z}}S_i^n\Delta x\Delta t. \end{aligned}$$

Hypothese for this poster : linear pressure $p(\tau) = a\tau$, $a < 0$.

Sketch of the proof - Tools

- (I_1) $\|\Delta\tau^n\|^2_{L^2_x} \le \frac{2}{|a|}\phi_\varepsilon^n$
- $(C-S)$ Cauchy-Schwarz
- (Y) Young
- (IBP_x) Discrete space integration by part

$$(I_2)\begin{cases} \|\delta_x X^n\|_{L^2_x} \le \frac{1}{\Delta x}\|X^n\|_{L^2_x} \\ \|\delta_{xx}X^n\|_{L^2_x} \le \frac{4}{\Delta x^2}\|X^n\|_{L^2_x} \\ \|\tilde{\delta}_x X^n\|_{L^2_x} \le \frac{2}{\Delta x}\|X^n\|_{L^2_x} \end{cases}$$

Sketch of the proof - Step 3

Estimation of different terms :

- $$\sum_{n=0}^{N}\sum_{i\in\mathbb{Z}}R_i^n\Delta x\Delta t \le \varepsilon^2\sum_{n=0}^{N}\|\Delta u^{n+1}\|_{L^2_x}\|\delta_t\overline{u}^{n+\frac12}\|_{L^2_x}\Delta t$$

$$(C-S)\quad + \varepsilon^2\Delta t\sum_{n=0}^{N}\|\Delta u^{n+1}\|_{L^2_x}\|\delta_{tx}p(\overline{\tau})^{n+\frac12}\|_{L^2_x}\Delta t$$

$$+ \frac{\lambda}{2}\varepsilon^2\Delta x\sum_{n=0}^{N}\|\Delta u^{n+1}\|_{L^2_x}\|\delta_{xx}\overline{u}^n\|_{L^2_x}\Delta t,$$

$$(H)\quad \le \left(1 + \Delta t + \frac{\lambda}{2}\Delta x\right)\varepsilon^2\sqrt{K}\sum_{n=0}^{N}\|\Delta u^{n+1}\|_{L^2_x}\Delta t,$$

$$(Y) \le \theta\sum_{n=0}^{N}\|\Delta u^{n+1}\|^2_{L^2_x}\Delta t + \frac{KT}{4\theta}\left(1 + \Delta t + \frac{\lambda}{2}\Delta x\right)^2\varepsilon^4$$

- Example of a term from $\sum_{n=0}^{N}\sum_{i\in\mathbb{Z}}S_i^n\Delta x\Delta t$:

$$-\frac{\lambda}{2}a\Delta x\sum_{n=0}^{N}\sum_{i\in\mathbb{Z}}\Delta\tau_i^{n+1}\delta_{xx}\Delta\tau_i^n\Delta x\Delta t$$

$$= -\frac{\lambda}{2}a\Delta x\Delta t\sum_{n=0}^{N}\sum_{i\in\mathbb{Z}}\delta_t\Delta\tau_i^{n+\frac12}\delta_{xx}\Delta\tau_i^n\Delta x\Delta t$$

$$(IBP_x)\quad 0 \ge \Big\{ -\frac{\lambda}{2}a\Delta x\sum_{n=0}^{N}\sum_{i\in\mathbb{Z}}\Delta\tau_i^n\delta_{xx}\Delta\tau_i^n\Delta x\Delta t,$$

$$\le -\frac{\lambda}{2}a\Delta x\Delta t\sum_{n=0}^{N}\sum_{i\in\mathbb{Z}}\Big(\delta_x\Delta u_i^n$$

$$(H_\varepsilon^\Delta) \text{ and } (P^\Delta)\quad + \frac{\lambda}{2}\Delta x\delta_{xx}\Delta\tau_i^n\Big)\delta_{xx}\Delta\tau_i^n\Delta x\Delta t,$$

$$\le 2\lambda|a|\frac{\Delta t}{\Delta x^2}\sum_{n=0}^{N}\|\Delta u^n\|_{L^2_x}\|\Delta\tau^n\|_{L^2_x}\Delta t$$

$$(C-S) \text{ and } (I_2)\quad + 4\lambda^2|a|\frac{\Delta t}{\Delta x^2}\sum_{n=0}^{N}\|\Delta\tau^n\|^2_{L^2_x}\Delta t,$$

$$(Y) \text{ and } (I_1)\quad \le |a|\frac{\Delta t}{\Delta x^2}\sum_{n=0}^{N}\|\Delta u^n\|^2_{L^2_x}\Delta t$$

$$+ 10\lambda^2\frac{\Delta t}{\Delta x^2}\sum_{n=0}^{N}\phi_\varepsilon^n\Delta t.$$

Conclusion of the proof

$$\begin{aligned} \phi_\varepsilon^{N+1} \le \phi_\varepsilon^0 + \alpha\|\Delta u^0\|^2_{L^2_x} + \beta\varepsilon^4 + \gamma\sum_{n=0}^{N}\phi_\varepsilon^n\Delta t \\ + \delta_1\sum_{n=0}^{N}\|\Delta u^{n+1}\|^2_{L^2_x}\Delta t + \delta_2\sum_{n=0}^{N}\|\Delta u^n\|^2_{L^2_x}\Delta t \end{aligned}$$

- Control terms in Δu with a negative constant δ.
- Use a discrete Gronwall lemma.

References

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