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Tensor Empirical Interpolation Method for multivariate functions

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Introduction

- We consider a multivariate function $f(x_1, \dots, x_d)$ defined over $\prod_{i=1}^{d} I_i$. We focus on tensors and/or low-rank tensors for the approximation of f in a separated form:

$$f(x_1, x_2, \dots, x_d) \approx \sum_{k=1}^{K} \prod_{i=1}^{d} \varphi_i^k(x_i) = \sum_{k=1}^{K} \varphi_1^k(x_1) \dots \varphi_d^k(x_d)$$

where, $\varphi_i^k(x_i)$, $i \in \{1, \dots, d\}$ are univariate functions.

- We propose an algorithm for the decomposition of multivariate functions using the Empirical Interpolation Method (EIM)[(1)].

[1] M. Barrault, Y. Maday, N. C. Nguyen, and A. T. Patera, *An empirical interpolation method: application to efficient reduced-basis discretization of partial differential equations*, Comptes Rendus Mathématique, 339(9) : 667-672, 2004.

Tensor Empirical Interpolation Method (TEIM) for bivariate functions[(2)]

- $f : I \times J \to \mathbb{R}$ a uniformly continuous function, m and n two fixed integers.

1. First step: EIM in x direction: "x space variable and $y \in J$ parameter"
   - $\tilde{y}_k = \arg\max_{y \in J} \| f(.,y) - \mathcal{I}_x^{(k-1)} f(.,y) \|_{L_x^\infty(I)}$
   - $x_k = \arg\max_{x \in I} | f(x, \tilde{y}_k) - \mathcal{I}_x^{(k-1)} f(x, \tilde{y}_k) |$
   - The basis function

$$q_k^{(k)}(x) = \frac{f(x, \tilde{y}_k) - \mathcal{I}_x^{(k-1)} f(x, \tilde{y}_k)}{f(x_k, \tilde{y}_k) - \mathcal{I}_x^{(k-1)} f(x_k, \tilde{y}_k)}.$$

   - A slight modification of the original EIM: update all the previous basis functions

$$q_i^{(k)}(x) = q_i^{(k-1)}(x) - q_i^{(k-1)}(x_k) q_k^{(k)}(x) \quad \forall i \in \{1, k-1\}.$$

   - The rank-k interpolation operator $\mathcal{I}_x^{(k)} f$

$$\mathcal{I}_x^{(k)} f(x,y) = \sum_{i=1}^{k} f(x_i, y)\, q_i^{(k)}(x), \quad y \in J.$$

2. Second step: EIM in y direction: "y space variable and $x \in I$ parameter"
   a) A set of interpolation points : $\{y_1, \dots, y_n\}$
   b) A set of basis functions $\{s_1^{(n)}, s_2^{(n)}, \dots, s_n^{(n)}\}$
   c) The rank-n interpolation operator

$$\mathcal{I}_y^{(n)} f(x,y) = \sum_{i=1}^{n} f(x, y_i)\, s_i^{(n)}(y), \quad x \in I.$$

3. Third step: tensorized interpolation of f:

$$\mathcal{I}^{m,n} f(x,y) = \mathcal{I}_y^{(n)} \mathcal{I}_x^{(m)} f(x,y) = \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j)\, q_i^{(m)}(x) s_j^{(n)}(y).$$

[2] F. De Vuyst, A. Toumi, *A mixed EIM-SVD tensor decomposition for bivariate functions.* ArXiv preprint 2017.

Extension of the TEIM to the multidimensional case

- $f(x_1, \dots, x_d)$ uniformly continuous function on $\Omega = \prod_{i=1}^{d} I_d$, $d \geq 2$.

1. EIM algorithm "direction-by-direction": for each variable x_i, $1 \leq i \leq d$, we compute: Interpolation points: $\{x_i^j\}_{j=1,m_i}$, basis functions: $\{q_{i,j}^{(m_i)}(x_i)\}_{j=1,m_i}$ and $\mathcal{I}_{x_i}^{(m_i)} f$.
2. The tensorized interpolation of the function f

$$\mathcal{I}^{m_1,\dots,m_d} f(x_1, \dots, x_d) = \sum_{j_1=1}^{m_1} \dots \sum_{j_d=1}^{m_d} f(x_1^{j_1}, \dots, x_d^{j_d})\, q_{1,j_1}(x_1) \dots q_{d,j_d}(x_d),$$

Properties of TEIM

- The basis functions $q_{i,j}^{(m_i)}$, $1 \leq j \leq m_i$ satisfy the Lagrange property
- Reinforced interpolation property: for all $1 \leq i \leq d$

$$\mathcal{I}^{m_1,\dots,m_d} f(x_1, \dots, x_{i-1}, x_i^k, x_{i+1}, \dots, x_d) = f(x_1, \dots x_{i-1}, x_i^k, x_{i+1}, \dots, x_d),$$

- A priori error estimation : f^* best approximation of f, $L_i^{m_i}$ the Lebesgue constants

$$\| f - I^{m_1,\dots,m_d} f \|_{L^\infty} \leq \inf_{f^* \in W^{m_1,\dots,m_d}} \| f - f^* \|_{L^\infty} \left(1 + \prod_{i=1}^{d} L_i^{m_i} \right)$$

Complexity of the TEIM

- Numerical complexity: curse of dimensionality for the multivariate extension
  - First solution: Singular Value Decomposition (SVD)
  - Second Solution: Sparse collocation method

Numerical results

$$f(x,y) = x + y + xy + e^{-(x^2+y^2)} + \sin(3\pi y) - \sin(\pi x y^2 + \pi x e^{-y})$$

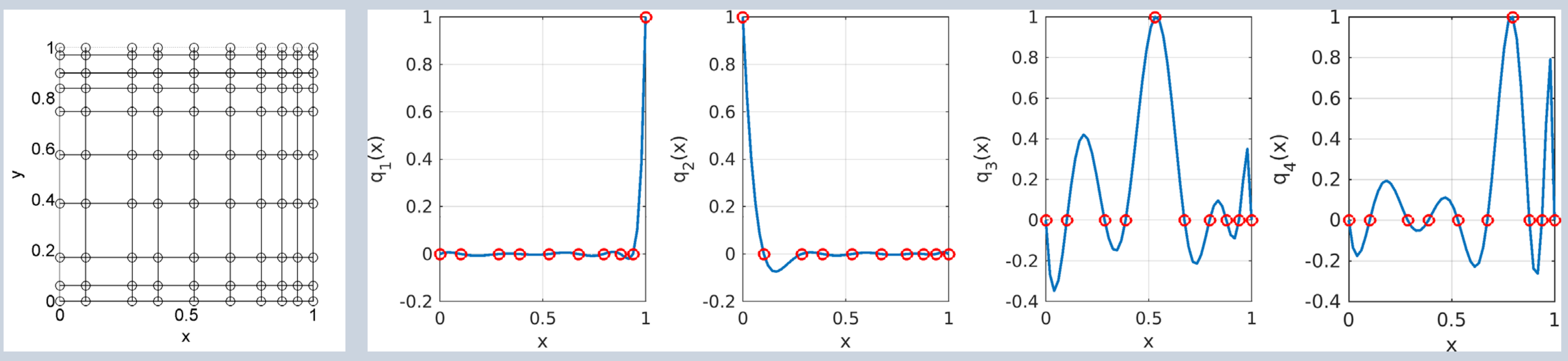


Figure: The first **100** interpolation points and the first four basis functions with the **10** TEIM points x_i.

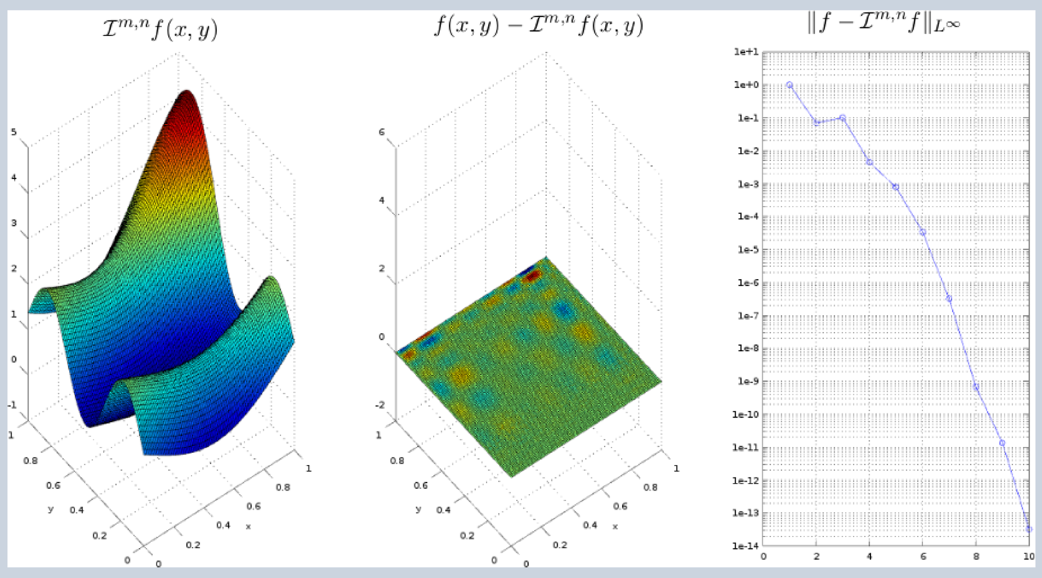


Figure: The TEIM interpolation

- The tensorized interpolation operator $\mathcal{I}^{10,10}$ (left).
- The difference between f and its approximation resulting from TEIM (middle).
- Convergence of the TEIM interpolation approach with respect to m (right)

- With the only rank-2 approximation we have a good reduced representation of the function f, with infinite-norm relative error of order 1%.

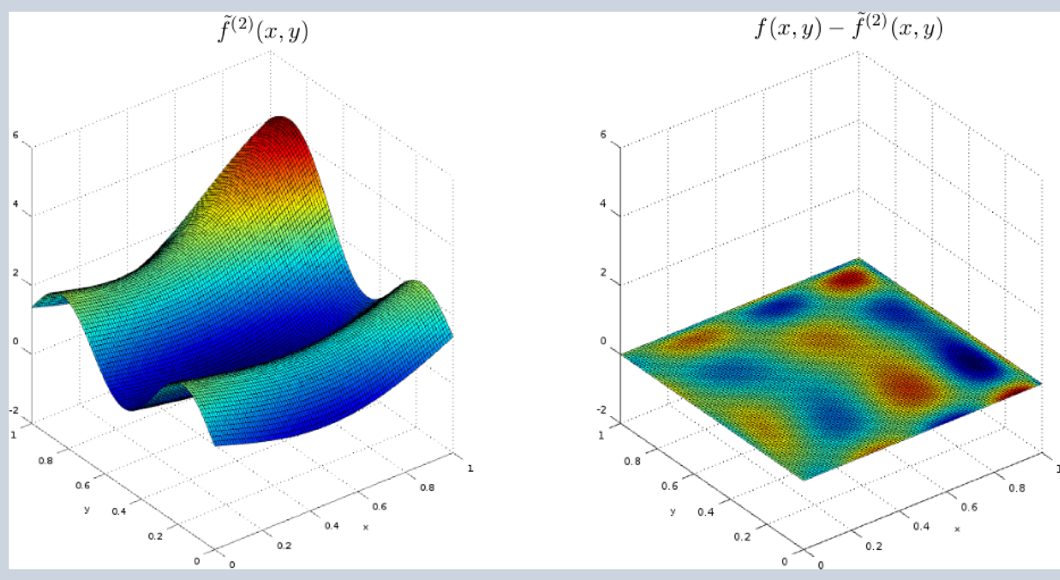


Figure: The SVD decomposition

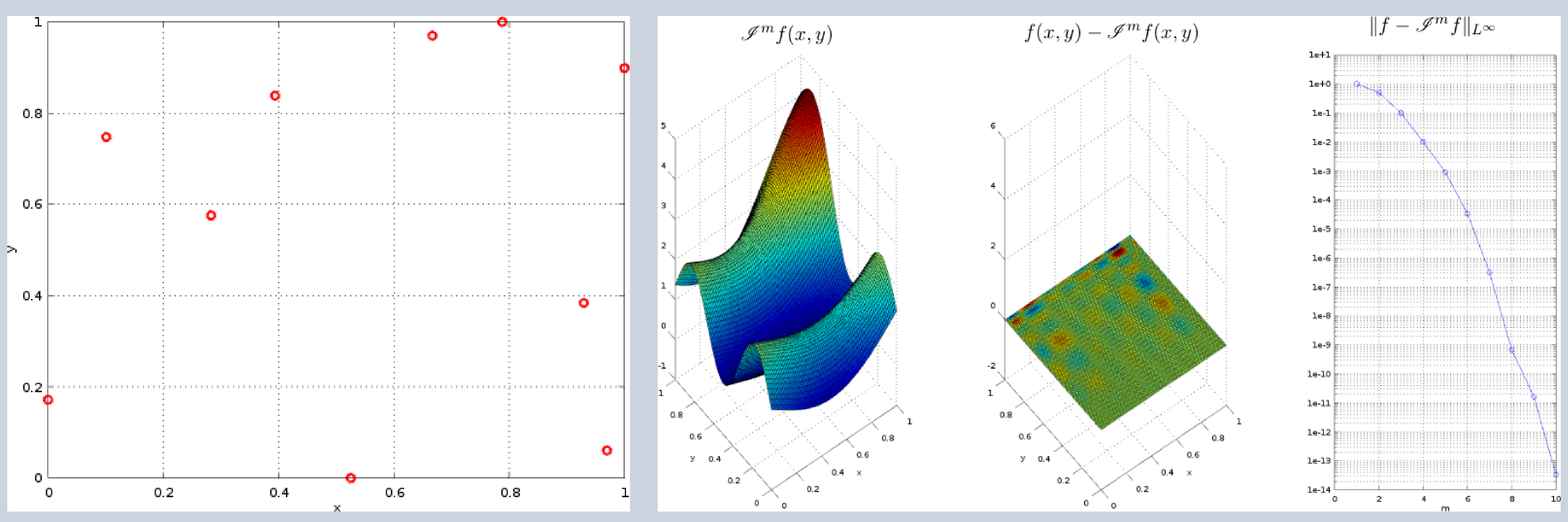
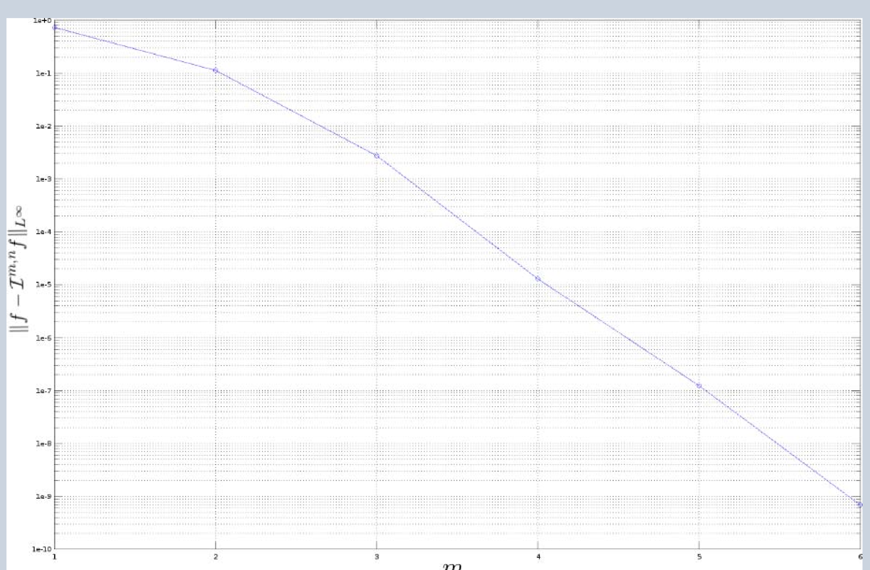


Figure: The sparse collocation points method



- $f = \sin(\pi x_1 x_2 x_3 x_4 x_5)$
- Numerical convergence confirm that even in the five dimensions case, the TEIM still have good properties.

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