Review of Numerical Schemes for the Classical Keller-Segel System - Nº 19

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Introduction and Previous Results

The movement of biological cells in response to chemical signals, also called as chemotaxis, was modelled by Keller-Segel in 1970. Although there are several models, we will focus on the classical one, which is given by the following equations in $\Omega \subset \mathbb{R}^n$:

$$
\begin{align*}
\partial_t u &= \Delta u - \alpha_1 \nabla \cdot (u \nabla v), & x \in \Omega, & t > 0, \\
\partial_t v &= \Delta v - \alpha_2 v + \kappa u v, & x \in \Omega, & t > 0,
\end{align*}
$$

(1)

where $u$ and $v$ represent the density of cells and chemical-signal, respectively. Recently, a lot of research in this subject has been done from an analytical point of view (see e.g. [1] and references therein). Although the number of results related to numerical analysis of (1) is much higher, we focus on this topic.

1. Global in time existence and boundedness of the solution has been shown if the initial data is small enough. For instance, for $n = 2$ one has:

**Theorem 1.** Suppose that $n = 2$ and $\Omega$ is a bounded domain with smooth boundary, that $u_0 \in C^2(\Omega)$ and $v \in L^2(\Omega)$ are non-negative, and that $(u_0, v_0)$ denotes the corresponding maximally extended classical solution of (1) in $\Omega \times (0, T_{\text{max}})$.

If $\int_{\Omega} u_0 < 4\pi$, then $(u(t), v(t))$ exist globally and satisfies

$$
\|u(t, \cdot)\|_{L^\infty(\Omega)} + \|v(t, \cdot)\|_{L^\infty(\Omega)} \leq C \quad \text{for all} \quad t > 0
$$

(2)

2. Energy law is important for a proof of Theorem 1 [1].

**Lemma 1.** If $(u, v)$ is a non-negative classical solution of (1) in $\Omega \times (0, T)$, then

$$
\frac{d}{dt} E(u(t), v(t)) = -D(u(t), v(t)),
$$

(3)

for all $t \in (0, T)$, with energy

$$
E(u, v) := \frac{1}{2} \int_\Omega |\nabla u|^2 + \frac{1}{2} \int_\Omega v^2 - \int_\Omega u v + \int_\Omega \ln u
$$

and the dissipation rate

$$
D(u, v) := \frac{1}{2} \int_\Omega |\nabla u|^2 - \int_\Omega u \frac{\partial v}{\partial t}.
$$

3. Blow-up phenomena have been detected for large initial data (e.g. $\int_{\Omega} u_0 > 4\pi$).

**Proposition 1.** Let be $(u, v)$ a solution of (1) in $\Omega \times (0, T)$ for some $t > 0$. If $\int_{\Omega} u_0 > 4\pi$ and $\int_{\Omega} v_0 > 4\pi$, then $u$ and $v$ are non-negative functions in their domain.

Discrete Energy Law and Positivity

Let us consider the following discrete counterpart of (3):

$$
E_m = \frac{1}{2} \int_\Omega |\nabla v|^2 + \frac{1}{2} \int_\Omega v^2 - \int_\Omega u v + \int_\Omega \sum_{m} \log (u_m),
$$

$$
D_m = \frac{1}{2} \int_\Omega |\nabla v|^2 + \frac{1}{2} \int_\Omega v^2 - \int_\Omega u v + \frac{1}{2} \int_\Omega |\Delta v|^2
$$

where we denote $\delta u_m = (u^{m+1} - u^m)/\Delta t$ for any sequence $(u^m)$, and let

$$
N_m = \sum_{m} \int_\Omega |\Delta v|^2 + \frac{1}{2} \int_\Omega |\Delta u|^2, \quad \gamma \in (0, 1).
$$

**Theorem 2.** Let us consider the schemes given by (4).

1. For $(r_1, r_2, r_3, r_4)$, the following energy law holds with $\gamma = 1$:

$$
d_{\text{tot}} E_m \leq -D_m - k N_m + k M_m,
$$

(5)

where $M_m \geq 0$ (numerical source) does not depend on $k$.

2. The case $(r_1, r_2, r_3, r_4) = (1, 1, 0, 0)$ is the only one with $M_m = 0$ (minimizes $M_m$).

3. The case $(r_1, r_2, r_3, r_4) = (1, 0, 1, 0)$ also satisfies (5) with $\gamma = 0$ (minimizes $N_m$).

4. If $r_1 = 1$ then $\mathbb{u}^{m+1}$ and $\mathbb{v}^{m+1}$ are non-negative solutions of (4) for all $m$.

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https://github.com/rrgalvan/keller-segel-schemes

References


