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A Well-balanced Finite Volume Scheme for Shallow Water Equations with Porosity - Nº22

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Framework

- Modelling emergent and rigid vegetation in open-channel with a single porosity-based shallow water model (SP) [1].
- New Godunov-type, finite volume, well-balanced and shock capturing scheme for SP.
- Comparison with other methods in the literature [2, 3] and first application.

Single porosity shallow water model

$$\partial_t W + \partial_x F + \partial_y G = S_b + S_\phi + S_\tau \tag{1}$$

with $W = \phi \begin{pmatrix} h \\ hu \\ hv \end{pmatrix}, F = \phi \begin{pmatrix} hu \\ hu^2 + \frac{g}{2}h^2 \\ huv \end{pmatrix}, G = \phi \begin{pmatrix} hv \\ huv \\ hv^2 + \frac{g}{2}h^2 \end{pmatrix},$

$$S_b = \phi \begin{pmatrix} 0 \\ -gh\partial_x b \\ -gh\partial_y b \end{pmatrix}, S_\phi = \begin{pmatrix} 0 \\ \frac{g}{2}h^2\partial_x\phi \\ \frac{g}{2}h^2\partial_y\phi \end{pmatrix}, S_\tau = -\begin{pmatrix} 0 \\ g\phi h\frac{\eta^2|U|}{h^{4/3}}u + \frac{1}{2}\frac{aC_d h|U|}{\phi}u \\ g\phi h\frac{\eta^2|U|}{h^{4/3}}v + \frac{1}{2}\frac{aC_d h|U|}{\phi}v \end{pmatrix}.$$

where

$$\begin{cases} h & : \text{depth of water}, \\ b & : \text{bed elevation}, \\ \eta & : \text{Manning's coefficient}, \\ C_d & : \text{drag coefficient}, \\ \phi & : \text{porosity}, \end{cases} \qquad \begin{cases} u, v & : \text{depth-averaged horizontal velocities}, \\ g & : \text{acceleration due to gravity}, \\ D & : \text{effective diameter of vegetation}, \\ a = \frac{1-\phi}{\pi D/4} & : \text{frontal area of vegetation}. \end{cases}$$

Numerical scheme

Well-balanced scheme : we solve (1) in two following steps:

Convection:

$$\begin{cases} \partial_t W + \partial_x F + \partial_y G = S_b + S_\phi, \\ W(x,0) = W^n. \end{cases} \tag{2}$$

Friction and Drag:

$$\begin{cases} \partial_t W = S_\tau, \\ W(x,0) = W^{n+1/2}. \end{cases} \tag{3}$$

FV scheme for (2): $W_j^{n+1/2} = W_j^n - \dfrac{\Delta t}{|C_j|} \sum_{\Gamma_{jk}\subset\partial C_j} |\Gamma_{jk}| \mathcal{F}(W_j^n, W_k^n; b_j, b_k)$.

Rotational invariance of the flux $\Rightarrow$ numerical flux derived from the 1D system (4).

$$\begin{cases} \partial_t W + \partial_x F = S_x, \\ W(x,0) = \begin{cases} W_L & \text{if } x < 0, \\ W_R & \text{if } x > 0. \end{cases} \end{cases} \tag{4}$$

Riemann ($\mathcal{R}$.) problem with $S_x(W,b) = (0, \frac{g}{2}h^2\partial_x\phi - g\phi h\partial_x b, 0)$ and $W_{L,R}$ the projected states $W_{j,k}^n$ on Γ_{jk}.

Approximation by simple-solver $W_{\mathcal{R}}(x/t)$ of (4):

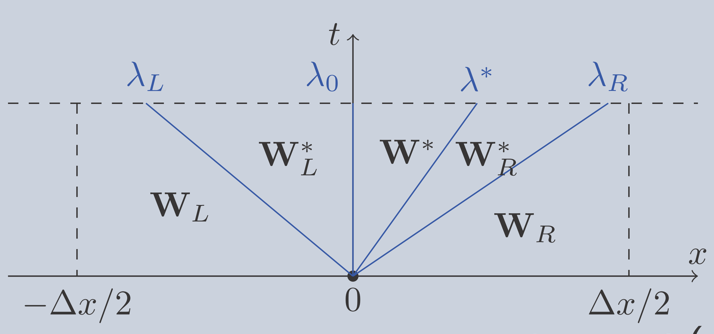

Figure: Four-wave approximate solution of (4).

First order Godunov-type scheme :

$$W_j^{n+1/2} = W_j^n - \frac{\Delta t}{\Delta x}\left(F^L_{j+1/2} - F^R_{j-1/2}\right)$$

$$\begin{cases} F^L = F(W_L) + \lambda_L(W_L^* - W_L) + \lambda_-^*(W^* - W_L^*), \\ F^R = F(W_R) - \lambda_R(W_R - W_R^*) - \lambda_+^*(W_R^* - W^*). \end{cases}$$

Determination of intermediate states:

- *integral consistency* condition:

$$\frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} W_{\mathcal{R}}\left(\frac{x}{\Delta t}\right)dx = \frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} W\left(\frac{x}{\Delta t}\right)dx \tag{5}$$

- HLL state for homogeneous $\mathcal{R}$. problem:

$$W^{HLL} = \frac{\lambda_R W_R - \lambda_L W_L}{\lambda_R - \lambda_L} - \frac{F(W_R) - F(W_L)}{\lambda_R - \lambda_L}$$

- $\mathcal{R}$. invariants across contact discontinuities:

$$\phi hu = \text{cst.},\ u^2/(2g) + h + b = \text{cst.},\ v = \text{cst.} \tag{6}$$

- (5) and 1st invariant in (6) lead to:

$$(\phi h)^{HLL} = \alpha\phi_R h_R^* + (1-\alpha)\phi_L h_L^*,\quad \alpha = \frac{\lambda_R}{\lambda_R - \lambda_L}, \tag{7}$$

$$(\phi hu)^{HLL} = q^* - \frac{\Delta x\,\overline{S_x}}{\lambda_R - \lambda_L},\quad \overline{S_x} = \frac{1}{|C|}\int_C S_x dxdt. \tag{8}$$

- (7) and 2nd invariant in (6) lead to:

$$h_L^* = \frac{(\phi h)^{HLL} + \alpha\phi_R(b_R - b_L)}{\alpha\phi_R + (1-\alpha)\phi_L},$$
$$h_R^* = \frac{(\phi h)^{HLL} - (1-\alpha)\phi_L(b_R - b_L)}{\alpha\phi_R + (1-\alpha)\phi_L}.$$

- (5) and 3rd invariant (6) yield:

$$\lambda^* = \frac{\phi_L h_L(u_L - \lambda_L) + \phi_R h_R(u_R - \lambda_R)}{2(\phi h)^*} + \frac{\lambda_L\phi_L h_L^* + \lambda_R\phi_R h_R^*}{2(\phi h)^*}$$

- steady states at rest in (8) yield:

$$\Delta x\,\overline{S_x} = g\left(\frac{h_L h_R}{2}(\phi_R - \phi_L)\right) - g\left(\frac{\phi_L h_L + \phi_R h_R}{2}(b_R - b_L)\right)$$

Semi-implicit scheme for (3): $(\phi hU)_j^{n+1} = \dfrac{(\phi hU)_j^{n+1/2}}{1 + \Delta t\left(\frac{\eta^2 g|U_j^n|}{(h_j^{n+1})^{4/3}} + \frac{1}{2}\frac{aC_d|U_j^n|}{\phi_j^2}\right)}$

Numerical experiment

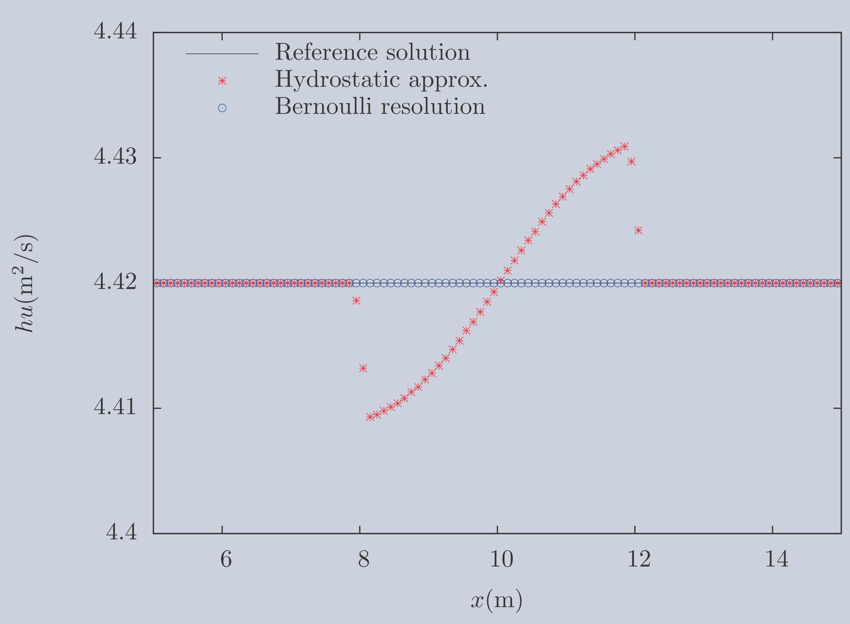

Figure: Steady sub-critical flow over a bump.

- well-known test case of SW model.
- Bernoulli relation in (6):

$$\frac{u_L^{*2}}{2g} + h_L^* + b_L = \frac{u_R^{*2}}{2g} + h_R^* + b_R.$$

- hydrostatic approximation yields:

$$h_L^* + b_L = h_R^* + b_R.$$

- as efficient as the PorAS scheme [2] for this type of regular solution.

- test case from [3], exact solution known.
- complex structure: 1-rarefaction wave, stationary discontinuity, 1-rarefaction wave then 2-shock wave.
- hydrostatic approximation: wrong solution unlike Bernoulli resolution.
- PorAs scheme can have difficulties to estimate shock waves speed.

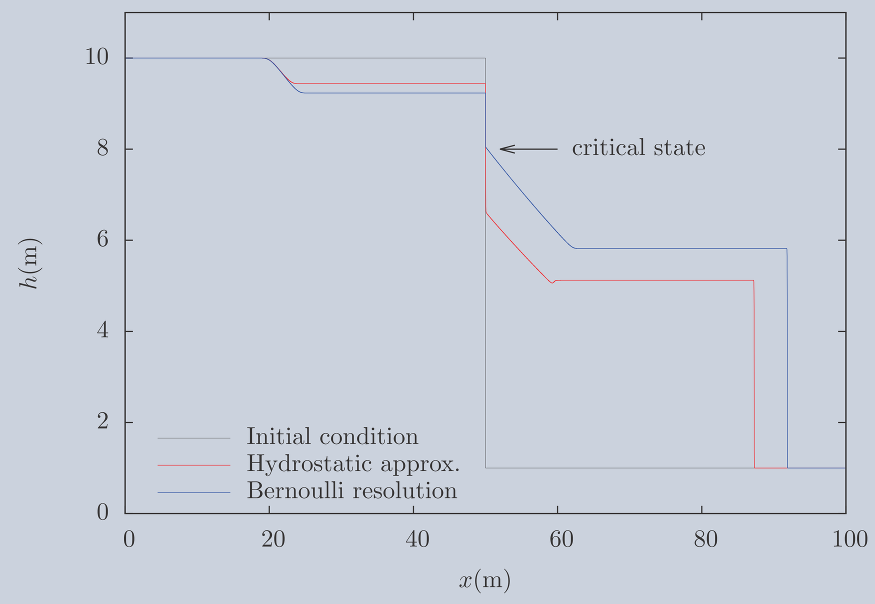

Figure: Dambreak with large porosity discontinuity.

Longitudinal transition from meadow to wood in a laboratory flume [4]:

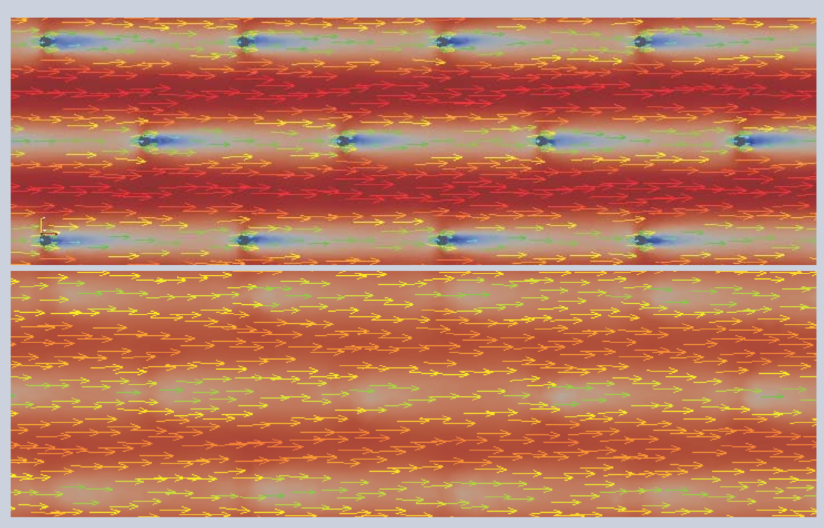

Figure: Velocity field and unit discharge in wood region: SW model (top) and SP model (bottom).

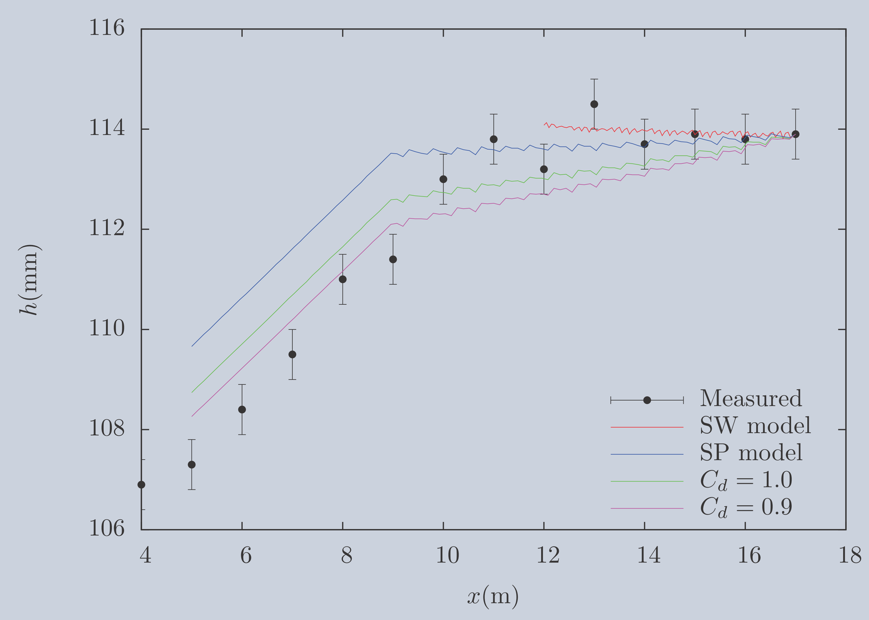

Figure: Water depth around the transition.

- emergent, rigid and circular vegetation distributed in staggered rows.
- expensive computation cost with SW.
- SP capture macroscopic behaviours.

- SP capture qualitative behaviour.
- agreement in wood region, $C_d = 1.2$.
- better compromise with $C_d = 1$.

Conclusions & Forthcoming Research

- augmented HLLC scheme well-balanced, positivity preserving and shock capturing.
- further test cases have to be implemented.

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