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3 Spectral Variational Multi-Scale methods for two-dimensional convection-diffusion problems

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Introduction

- Method developed in [1] for 1D elliptic problems and generalized in [2] to parabolic problems.
- In [3], computation of the stabilization coefficients in the problems in [1,2].
- Consider 2D elliptic problems with Dirichlet boundary conditions.
- Eigenpairs of the convection-diffusion operator. Two types of meshes: squares and triangles.
- Stabilization coefficients are computed in an off-line phase.
- Numerical tests to compare our results with those obtained using orthogonal subscales [4].

Abstract formulation

Variational elliptic problem,

$$B(U,V) = l(V), \quad \forall V \in X, \tag{1}$$

where X is a Hilbert space, B a bilinear bounded and coercive form and $l \in X'$, the topological dual of X.
Standard VMS reformulation of problem (1),

$$B(U_h,V_h) + B^S(U_h,V_h) = l(V_h), \quad \forall V_h \in X_h,$$

where $X = X_h \oplus \tilde{X}$.

Theorem *Let us assume that there exists a complete sub-set $\{\hat{z}_{js}^{(K)}\}_{j,s\in\mathbf{N}}$ on $\tilde{X}_K$ formed by eigenfunctions of the operator $\mathcal{L}_K$, which is an orthonormal system in $L^2_{p_K}(K)$ for some weight function $p_K \in C^1(\bar{K})$. Then,*

$$\tilde{U}_K = \sum_{j=1}^{\infty}\sum_{s=1}^{\infty} \beta_{js}^K \langle R(U_h), p_K \hat{z}_{js}^{(K)}\rangle \hat{z}_{js}^{(K)}, \quad \text{with } \beta_{js}^K = (\lambda_{js}^K)^{-1},$$

where λ_{js}^K is the eigenvalue of $\mathcal{L}_K$ associated to $\hat{z}_{js}^{(K)}$.

Formulation of the method

$$B(U_{h,M_1,M_2},V_h) + B^S(U_{h,M_1,M_2},V_h) = l(V_h), \quad \forall V_h \in X_h,$$

where BS models the effect of subgrid scales on resolved ones.

Application to two-dimensional convection-diffusion problem

Variational formulation of the stationary convection-diffusion problem:
Find $U \in H_0^1(\Omega)$ such that,

$$(\mathbf{a}\cdot\nabla U, V) + \mu(\nabla U, \nabla V) = (f,V) \quad \forall V \in H_0^1(\Omega).$$

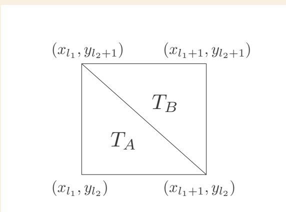

Figure: The two types of right isosceles triangles considered.

- Eigenpairs of the convection-diffusion operator can be computed in terms of that of the Laplacian.
- We consider two type of meshes: squares and isosceles right triangles such as those in the figure.

Computation of the stabilization coefficients

$$B^S(U_{h,M_1,M_2},V_h) \simeq \sum_{K\in\mathcal{T}_h} \tau^{(K)}_{M_1,M_2}(\mathbf{a}_K\cdot\nabla U_{h,M_1,M_2}, \mathbf{a}_K\cdot\nabla V_h)_K, \text{ where}$$

$$\tau^{(K)}_{M_1,M_2} = \frac{1}{|K|}\sum_{j=1}^{M_1}\sum_{s=1}^{M_2}\beta_{js}^K\left(\int_K p_K\hat{z}_{js}^{(K)}\right)\left(\int_K \hat{z}_{js}^{(K)}\right).$$

Numerical results

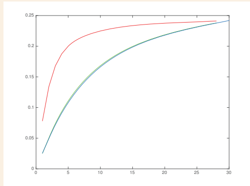

Figure: Comparison between stabilized 1D and 2D coefficients.

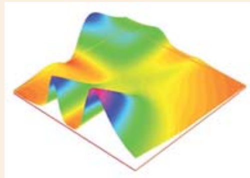

Figure: Stabilized coefficients 2D case with $P_1, P_2 \in (-10, 10)$.

Figure: Stabilized coefficient for anisotropic test $P_1, P_2 \in (-5, 5)$.

Figure: Stabilized solution anisotropic test $P_1, P_2 \in (-5, 5)$

Table: Errors of anisotropic test when $P_1, P_2 \in (-5, 5)$

N	L^2_a	L^2_o	H^1_a	H^1_o	m_a	m_o
50	5.080e-3	9.972e-3	1.5684	1.7546	0.1317	0.1720
100	1.717e-3	5.15e-3	1.045	1.3137	0.065	0.1149
200	4.83e-4	2.18e-3	0.5177	0.7325	0.025	0.0604

Conclusions and perspectives

- Extension to 2D systems the VMS-spectral method for elliptic problems.
- Application to the stationary convection-diffusion problem.
- For piecewise affine finite element discretizations, this method can be cast as a standard VMS method with approximate stabilized coefficients.
- For 2D convection-diffusion equations these coefficients can be computed in an off-line phase, and then interpolated in the on-line phase.
- In order to have a exact decomposition of the 2D space, subscales on the element boundaries need to be considered.
- Apply the method to 2D evolutive problems.

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