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5. An inverse problem governed by the Lamè System: tumor identification

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Introduction

Elastography in Medicine: is a field whose main goal is to reconstruct the internal distribution of a body from the measurements on the contour of the responses to induced elastic waves

Important application: Non-invasive technique for detection and characterization of tumors (tumor stiffness $\sim$ 5–28 times bigger than for healthy cells) and the differentiation of benign and malignant tumors ([3])

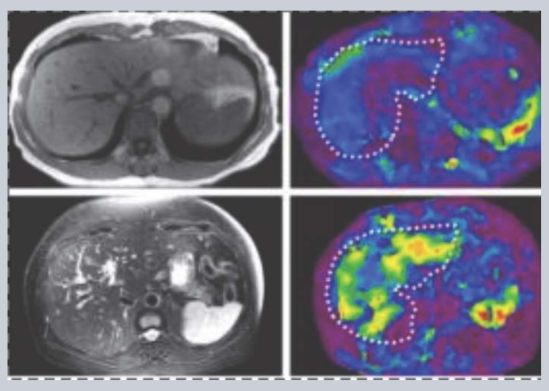

Figure: Left : Tradictional black and white image - Right: MRE scan

Mathematically, the elastography problem can be written as an **inverse problem**.
For a **Lamé system**: Find λ and μ such that the solution of

$$(IP)\quad \begin{cases} u_{tt} - \mathbf{div}(\mu\nabla u + \lambda(\nabla\cdot u)Id) = 0 & \text{in } (0,T)\times\Omega, \\ +\text{boundary} + \text{initial conditions} \end{cases}$$

satisfies the additional condition

$$(\mu\nabla u + \lambda(\nabla\cdot u)Id)\cdot\nu = \Upsilon \text{ on } S\times(0,T)(S\subset\partial\Omega)$$

Our Problem

We analyze a problem governed by the **linear elasticity system** in $\Omega\subset\mathbb{R}^N$, with $N = 2, 3$.

We know Υ on $S\times(0,T)$ $(S\subset\partial\Omega)$

The inverse problem: Find $A = A(x)$ such that the solution of

$$(IP)\quad \begin{cases} u_{tt} - \mathbf{div}(A(x)e(u)) = f(x,t) & \text{in } \Omega\times(0,T), \\ u(x,t) = 0 & \text{on } \partial\Omega\times(0,T), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) & \text{in } \Omega, \end{cases}$$

with $e(u) = \frac{1}{2}(\nabla u + \nabla u^t)$ and satisfing the additional condition

$$Ae(u)\cdot\nu = \Upsilon \text{ on } S\times(0,T).$$

The direct problem: Find u from the data A, f u_0 y u_1 and, later, compute Υ.

$$(DP)\quad A, f, u_0 \text{ and } u_1 \to \Upsilon.$$

Standard theory.

The inverse problem: Find A from f, u_0, u_1 and Υ.

$$(IP)\quad f, u_0, u_1 \text{ and } \Upsilon \to A.$$

Much more complicated (ill-posed)
Interesting questions: Uniqueness, stability, reconstruction, ...

Reformulation: A direct problem

We reformulate the problem as an **optimal design problem**:

$$I(A) = \frac{1}{2}\int_0^T \|Ae(u)\cdot\nu|_S - \Upsilon\|^2\,dt$$

$$(ODP)\quad \min_A I(A)$$

subject to
$$\begin{cases} u_{tt} - \mathbf{div}(A(x)e(u)) = f(x,t) & \text{in } \Omega\times(0,T), \\ u(x,t) = 0 & \text{on } \partial\Omega\times(0,T), \\ u(x,0) = u_0,\ u_t(x,0) = u_1(x) & \text{in } \Omega, \end{cases}$$

A sol. of (ODP) and $I(A) = 0 \Leftrightarrow A$ sol. (IP)

Framework

- $\alpha, \beta\in\mathbb{R}$ and $0 < \alpha < \beta$,
- $A = \{A_{ijkl}\}_{1\le ijkl\le N} \in \mathbb{M}(\alpha,\beta,\Omega)\cap\mathbb{BV}(\Omega)$, with
$$\mathbb{M}(\alpha,\beta,\Omega) = \begin{cases} A(x) = A(x)^T, \quad A(x)\Lambda\cdot\Lambda \ge \alpha|\Lambda|^2, \quad |A(x)\Lambda| \le \beta|\Lambda| \\ \text{a.e. in } \Omega \text{ for all symmetric } \Lambda\in\mathbb{R}^{N\times N} \end{cases}$$
- $u_0 = 0$, $u_1\in H_0^1(\Omega)^N$, $f, \dfrac{\partial f}{\partial t}\in L^2(\Omega\times(0,T))^N$,
- $\Upsilon\in L^2(S\times(0,T))^N$.

Theorem: A priori information: A uniformly bounded in BV, [1]

Let us set
$$\mathbb{K}(R) = \{\, A\in\mathbb{M}(\alpha,\beta,\Omega)\cap\mathbb{BV}(\Omega) : TV(A)\le R\,\}$$
and let us assume that the data of the problem satisfy (at least) the hypotheses described before. Then, for any $R > 0$, the extremal problem

$$\begin{cases} \text{Minimize}_{A\in\mathbb{K}(R)}\ I(A) & \text{subject to} \\ u_{tt} - \mathbf{div}(A(x)e(u)) = f(x,t) & \text{in } \Omega\times(0,T), \\ u(x,t) = 0 & \text{on } \partial\Omega\times(0,T), \\ u(x,0) = 0,\ u_t(x,0) = u_1(x) & \text{in } \Omega, \end{cases}$$

possesses at least one solution A_R^*.

Lemma

Assume that $A\in\mathbb{K}(R)$. There exists $\delta\in(0,1)$, only depending on α, β and R, such that, for any $h\in L^2(\Omega)^N$, the elliptic system

$$\begin{cases} -\mathbf{div}(Ae(w)) = h, & x\in\Omega \\ w = 0, & x\in\partial\Omega \end{cases}$$

possesses exactly one solution $w_h\in X_\delta := [D(\Delta), H_0^1(\Omega)^N]_{\delta,\infty}$, where $D(\Delta) = H_0^1(\Omega)^N\cap H^2(\Omega)^N$ is the domain of the Dirichlet Laplacian.
Furthermore, the mapping $h\mapsto w_h$ is linear and continuous, i.e.

$$\|w_h\|_{X_\delta} \le C(N,\Omega,\alpha,\beta,R)\,\|h\|_{L^2} \quad \forall h\in L^2(\Omega)^N.$$

Numerical experiments: 2-d Lamé system

- We consider $\Omega = B_2(0,1)\subset\mathbb{R}^2$, $S = \partial\Omega\cap\{x>0\}$.
- We use **Augmented Lagrangian algorithm**, completed with the **L-BFGS subalgorithm**.
- We use `FreeFem++ v 3.44` ([2]) completed with the library `NLopt`.
- We fix $\lambda(x)\equiv 1$ and minimize on $\alpha\le\mu(x)\le\beta$.

Simulations: one isolated tumor

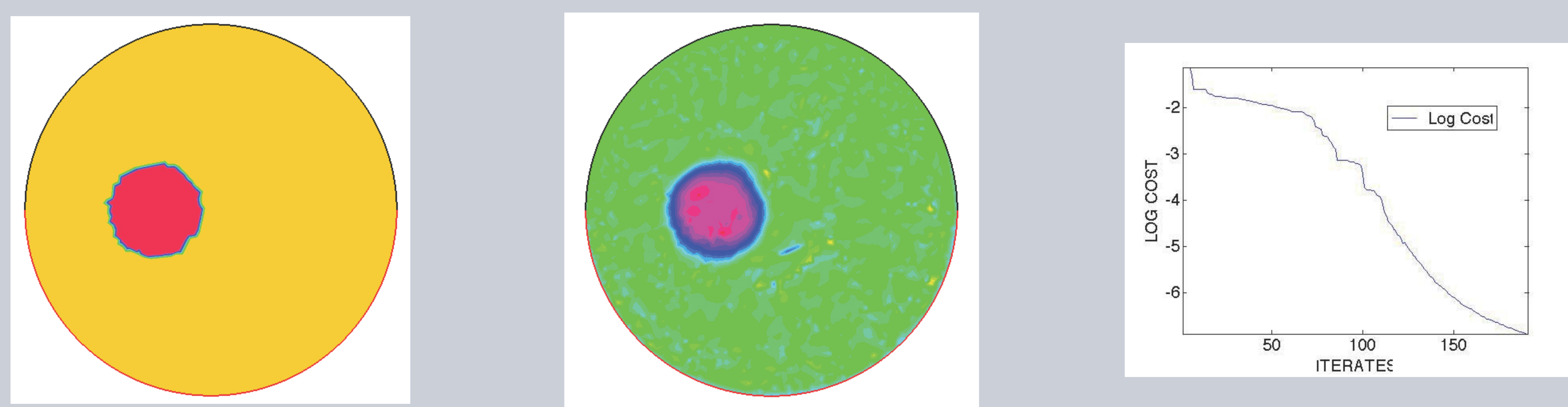

Figure: Left: Target tumor. Middle: Computed Tumor. Right: Decimal log of the cost vs. number of iterates

Simulations: two isolated tumors

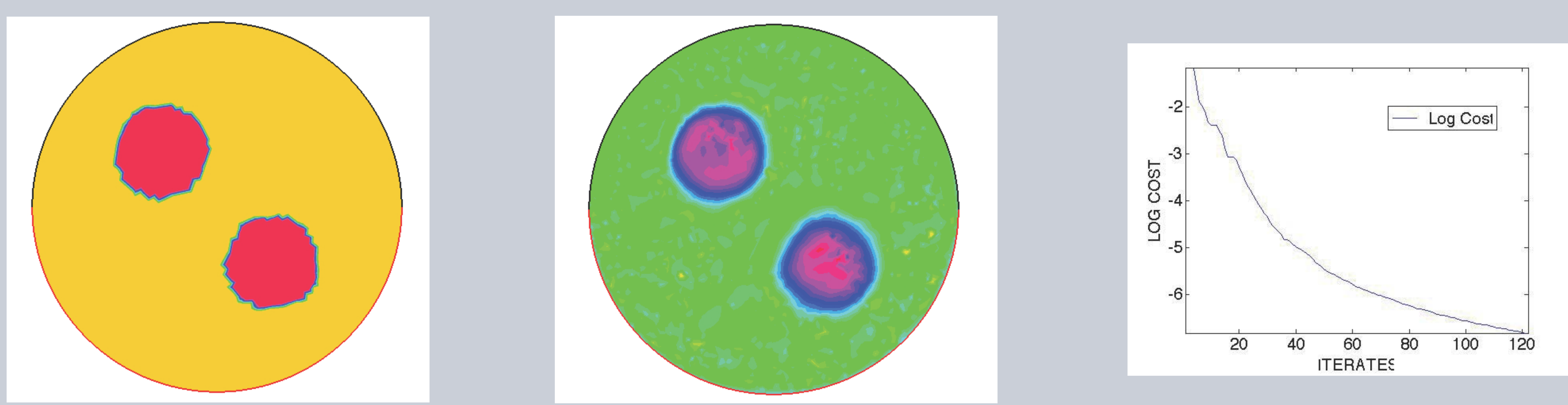

Figure: Left: Target tumor. Middle: Computed Tumor. Right: Decimal log of the cost vs. number of iterates

References

[1] Ferndez-Cara, E. and Maestre F. An inverse problem in elastography involving Lamè systems. Journal of Inverse and Ill-posed Problems, Published on-line. doi:10.1515/jiip-2017-0065.
[2] Hecht, F., *New development in FreeFem++*. J. Numer. Math., 20 (2012), 251-265.
[3] Manduca et al, *Magnetic resonance elastography: non-invasive mapping of tissue elasticity.* Med Imagen Anal. 5 (2001) 237-254.