XVIII SPANISH-FRENCH SCHOOL JACQUES-LOUIS LIONS ABOUT NUMERICAL SIMULATION IN PHYSICS AND ENGINEERING
Las Palmas de Gran Canaria, 25-29 June 2018

6. Optimal Intraregional Location of Emissions in a Transboundary Pollution Dynamic Game

Javier de Frutos[1], Paula M. López-Pérez[1], Guiomar Martín-Herrán[2]

[1]IMUVA and Departamento de Matemática Aplicada, Facultad de Ciencias, Universidad de Valladolid
[2]IMUVA and Departamento de Economía Aplicada, Facultad de Ciencias Económicas y Empresariales, Universidad de Valladolid

frutos@mac.uva.es, paulamaria.lopez@uva.es, guiomar@eco.uva.es

Introduction

Standard dynamic models used in the literature to study different types of economic and environmental problems have considered just time dependence. A recent study ([1]) takes into account not just the time but the spatial dependence and considers agents that behave both, dynamically and strategically. The model considered is a J-player non-cooperative differential game, where a planar region Ω is subdivided in J subregions Ω_j, $j = 1, \dots, J$. The objective of player j is to maximize its own payoff, choosing the rate of pollutant emissions in subregion Ω_j. The focus of the present work is to extend this research, studying the capabilities of the same model to analyse the optimal intraregional distribution of emissions, and to characterize feedback Nash equilibria of the resulting differential game.

Model problem

Let $u_j(x,t)$ for $j = 1, \dots, J$, the emission rate of subregion Ω_j. The spatio-temporal dynamics of the stock of pollution $P(x,t)$ is given by the parabolic PDE

$$\frac{\partial P}{\partial t} = \nabla \cdot (k \nabla P) - cP + F(\boldsymbol{u}) \quad \text{in } \Omega,$$

where $\boldsymbol{u} = [u_1, \dots, u_J]^T$ is the vector of emission rates, $k = k(x)$ is a local diffusion coefficient, which is assumed to be a smooth function. The term $-cP$ is a natural decay of the pollutant and the source term can be written in the form

$$F(\boldsymbol{u}(x,t)) = \sum_{j=1}^{J} F_j(u_j(x,t)) \mathbf{1}_{\Omega_j}(x),$$

being F_j a given family of smooth functions for $j = 1, \dots, J$ and $\mathbf{1}_{\Omega_j}$ the characteristic function of subregion Ω_j. The dynamics is completed with an initial condition

$$P(x,0) = P_0(x) \quad \text{in } \Omega,$$

where P_0 is the initial distribution of the stock of pollution, and a boundary condition

$$\alpha(x) P(x,t) + k(x) \nabla P^T(x,t) \boldsymbol{n} = 0 \quad \text{on } \partial\Omega,$$

being $\alpha(x)$ a non-negative smooth function and $\boldsymbol{n}$ the normal vector exterior to Ω. Objective of player j is to choose the distributed control u_j, in order to maximize its payoff

$$J_j(u_1, \dots, u_J, P_0) = \int_0^{+\infty} \int_{\Omega_j} e^{-\rho t} G_j(u_1, \dots, u_J, P) \, dx dt,$$

where $\rho > 0$ is a given time-discount rate and G_j are the benefits from consumption net of environmental damages.

Extended discrete-space model

An extension of the discrete-space model studied in ([1]) is deduced as follows. Each subregion Ω_j is subdivided in N_j smaller subregions $\Omega_{j,l}$, for $j = 1, \dots, J$ and $l = 1, \dots, N_j$. Let $p_{j,l}(t)$ the averaged stock of pollution over subregion $\Omega_{j,l}$, and $v_{j,l}(t)$ the averaged emissions over $\Omega_{j,l}$, for $j = 1, \dots, J$ and $l = 1, \dots, N_j$,

$$p_{j,l}(t) = \frac{1}{|\Omega_{j,l}|} \int_{\Omega_{j,l}} P(x,t) \, dx, \quad v_{j,l}(t) = \frac{1}{|\Omega_{j,l}|} \int_{\Omega_{j,l}} u_j(x,t) \, dx.$$

Let $N = N_1 + \ldots + N_J$ the total number of subregions and h a biyective function that relates global and local order of subregions. The objective of player j is to maximize the space averaged payoff

$$\widetilde{J}_j(v_{j,1}, \dots, v_{j,N_j}, \boldsymbol{p}^0) = \int_0^{+\infty} e^{-\rho t} \sum_{l=1}^{N_j} |\Omega_{j,l}| \widetilde{G}_j(v_{j,l}, p_{j,l}) \, dt,$$

subject to the dynamics of the aggregated stock of pollution in each subregion, described by the following system of ODEs:

$$\dot{p}_{h(i_1)} = \frac{1}{|\Omega_{h(i_1)}|} \sum_{i_2=0,\, i_2 \neq i_1}^{N} k_{i_1 i_2}(p_{h(i_2)} - p_{h(i_1)}) - c_{h(i_1)} p_{h(i_1)} + F_{h(i_1)}(v_{h(i_1)}), \qquad i_1 = 1, \dots, N,$$

being $c_{h(i_1)}$ the averaged natural decay parameter and $(k_{i_1 i_2})_{1 \le i_1, i_2 \le N}$ the diffusion matrix, satisfying $k_{i_1 i_1} = -\sum_{i_2 \neq i_1} k_{i_1 i_2}$. The system is supplemented with the initial condition

$$p_{h(i_1)}(0) = \frac{1}{|\Omega_{h(i_1)}|} \int_{\Omega_{h(i_1)}} P_0(x) \, dx := p^0_{h(i_1)}, \qquad i_1 = 1, \dots, N,$$

where $\boldsymbol{p}^0 = [p_1^0, \dots, p_N^0]^T$.

A linear-quadratic specification

Some specifications (inspired in the literature of transboundary pollution dynamic games, [2]) are introduced to characterize the feedback Nash equilibria. Let $\beta_{j,l}$, $A_{j,l}$ and $\varphi_{j,l}$ constant parameters for all $j = 1, \dots, J$ and $l = 1, \dots, N_j$. Then,

$$F_{j,l}(v_{1,1}, \dots, v_{J,N_J}) := \beta_{j,l} v_{j,l}, \quad \widetilde{G}_j(v_{1,1}, \dots, v_{J,N_J}, \boldsymbol{p}) := v_{j,l}\left(A_{j,l} - \frac{v_{j,l}}{2}\right) - \frac{\varphi_{j,l}}{2} p_{j,l}^2, \quad \boldsymbol{p} := [p_1, \dots, p_N]^T, \quad v_{j,l} = v_{j,l}(\boldsymbol{p}), \quad |\Omega_{h(i_1)}| = |\Omega_{h(i_2)}|, \quad \forall\, i_1, i_2 = 1, \dots, N.$$

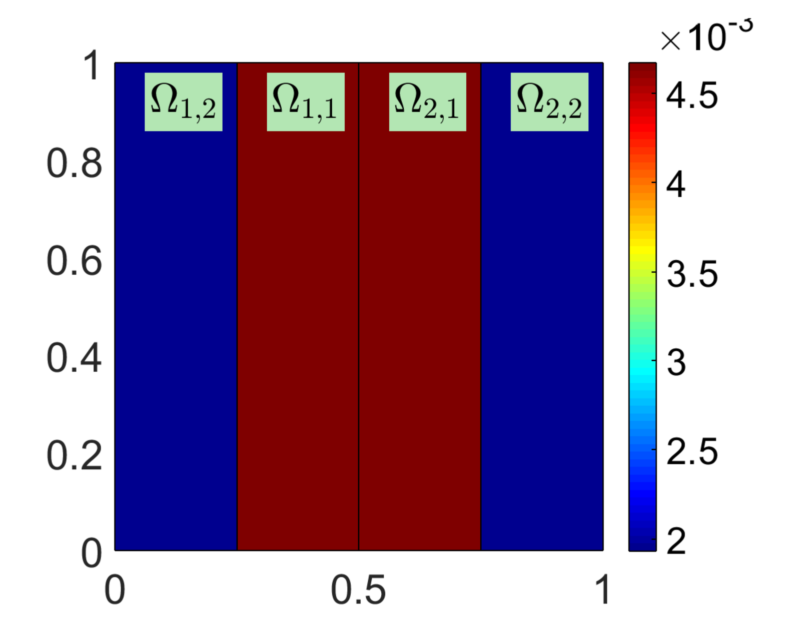
Numerical method and results

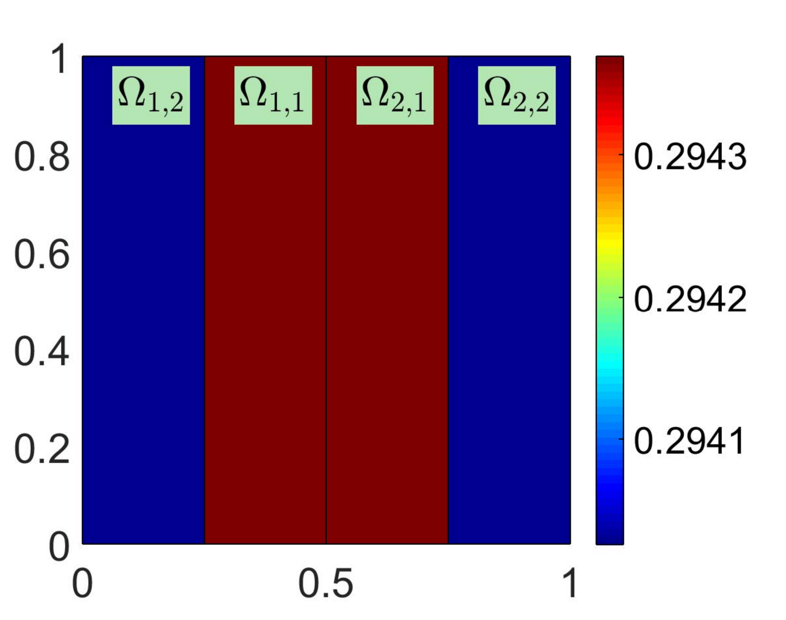
In order to characterize numerically the feedback Nash equilibria of the differential game, a numerical method in two steps is introduced. On the first step, a time-discrete version of the problem is considered, discretizing the averaged payoff by means of the rectangle rule and then using a forward Euler discretization of the dynamics. Secondly, a space-discretization is applied. The time-discrete value function is computed solving a system of Bellman equations, and the solution of this system is approximated using a collocation method based on tensorial product of linear splines and a fixed-point iteration.

Two numerical examples are presented. In both, a two-player (two-region) version of the described differential game is considered. Region Ω_1 is subdivided in two subregions, $\Omega_{1,1}$ and $\Omega_{1,2}$, and region Ω_2 is subdivided in two subregions too, $\Omega_{2,1}$ and $\Omega_{2,2}$. Each region optimally chooses the emission rates in the two subregions under its control. The parameters in the linear-quadratic specification are chosen equal in each subregion, $A_{j,l} = 0.5$, $\varphi_{j,l} = 1$, $\beta_{j,l} = 1$, $c_{j,l} = 0.01$, $\rho = 0.1$ for all $j, l = 1, 2$, and the measure of each subregion is considered $|\Omega_{j,l}| = 0.25$.

Neumann homogeneous boundary conditions

In the first example, homogeneous Neumann boundary conditions ($\alpha(x) = 0$ for all x in Ω) are considered all over the boundary of the domain. This means that there is no exchange of pollution with the exterior of Ω. The players are completely symmetrical in every respect, except in their geographical positions.

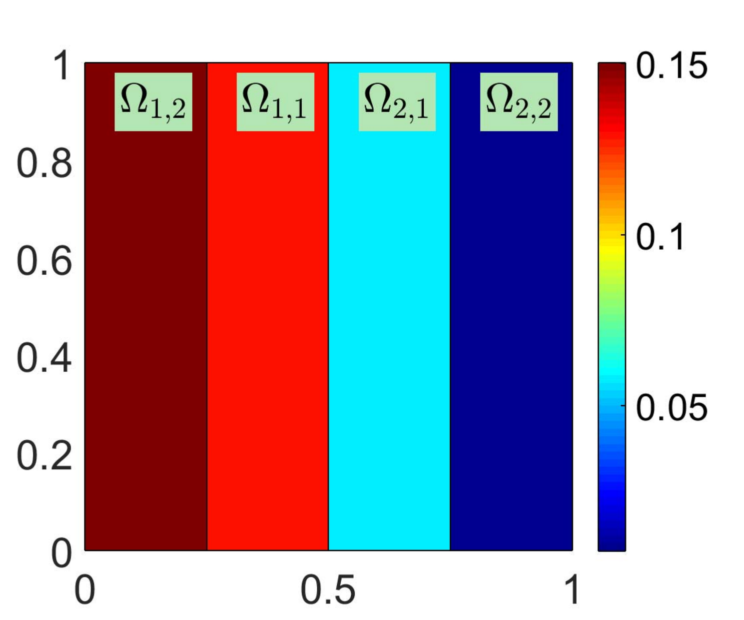


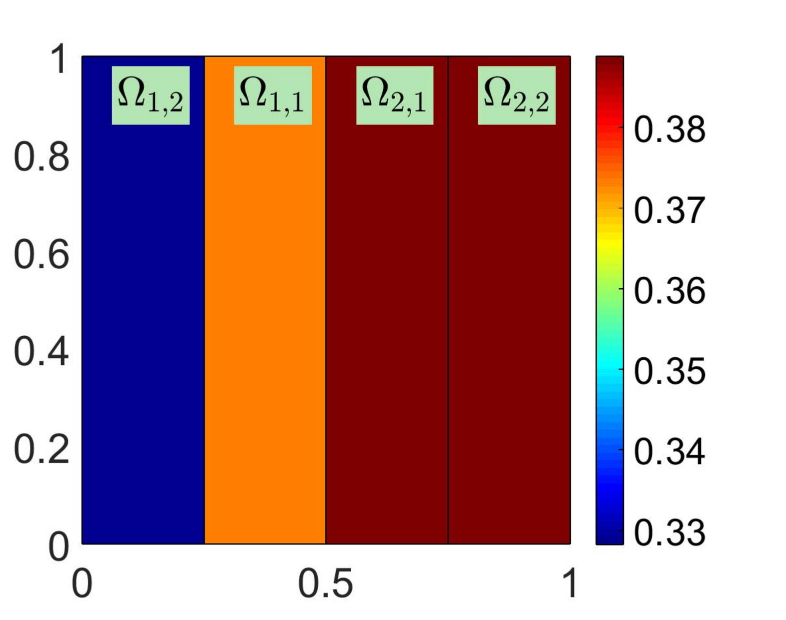


Figures above show that the two regions behave in a symmetrical way. Both emit at the greatest level in the subregion that shares a boundary with its neighbour. The left plot shows the equilibrium emission rates and the right one the steady-state levels of the stock pf pollution.

Adding Dirichlet homogeneous boundary conditions

In the second example, homogeneous Neumann boundary conditions are considered all over the boundary of the domain ($\alpha(x) = 0$), except in the upper boundary of $\Omega_{1,2}$ where an homogeneous Dirichlet boundary condition is imposed ($\alpha(x) = 1$). The exterior of Ω is assumed to have a lower concentration of pollutants than subregion $\Omega_{1,2}$.





The equilibrium emission strategies have lost the symmetric property observed in the first example, as it is shown on the Figures above. Region Ω_1 can benefit from the spatial position of $\Omega_{1,2}$ and emit in this subregion above the level of emissions in the other subregion. However, the long-run stock of pollution in subregion $\Omega_{1,2}$ is lower than in the other subregions. This can be explained because of the flow of pollution exiting $\Omega_{1,2}$ towards the exterior of Ω. This results cannot be reproduced in a transboundary pollution dynamic game with symmetric players if the spatial transport of pollution is neglected.

Conclusions

A J-player non-cooperative differential game that takes into account spatial dependence is studied. An extended discrete-space model is deduced (fitting the structure used in [3]) by means of aggregated variables, maintaining the three main features of the original formulation: the model is truly dynamic, the agents behave strategically and the model incorporates the spatial aspect. Some numerical results illustrate the optimal intraregional distribution of emissions of the pollutant in each subregion. One of the difficulties that it is necessary to confront is the high dimensionality of the Hamilton-Jacobi-Bellman system of equations that characterizes the feedback Nash equilibria.

Acknowledgements

This research was partially supported by Junta de Castilla y León under project VA024P17, co-financed by FEDER funds.

References

[1] J. De Frutos, G. Martín-Herrán, *Spatial Effects and Strategic Behaviour in a Multiregional Transboundary Pollution Dynamic Game*, Journal of Environmental Economics and Management (2017), `https://doi.org/10.1016/j.jeem.2017.08.001`

[2] S. Jørgensen, G. Martín-Herrán, G. Zaccour, *Dynamic Games in the Economics and Management of Pollution*, Environmental Modelling and Assessment 15, (2010), 433–467.

[3] K-G. Mäler, A. De Zeeuw, *The Acid Rain Differential Game*, Environmental and Resource Economics 12, (1998), 167–184.