Nonlinear subdivision in Uncertainty Quantification

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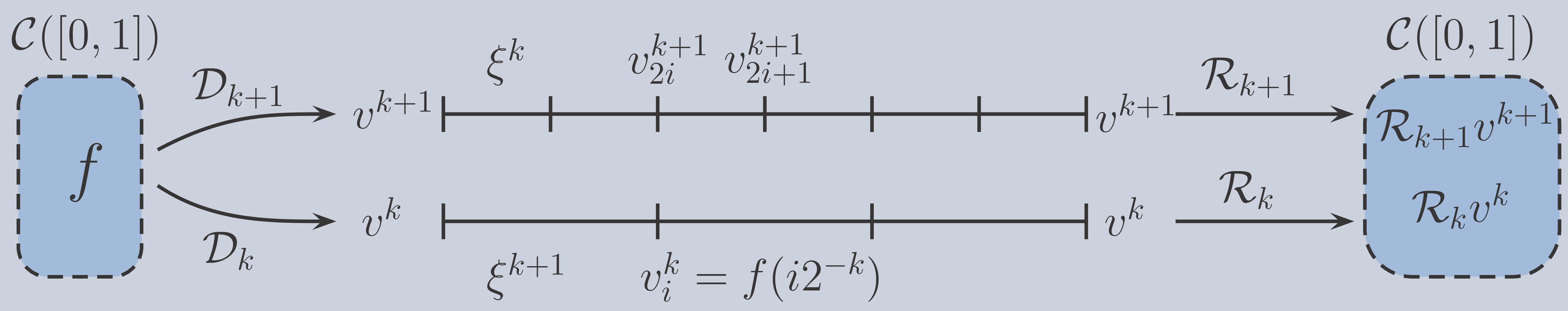
Abstract

This work is concerned about a non-intrusive method for Partial Differential Equations (PDEs) in Uncertainty Quantification (UQ), which is based on Harten's Multiresolution Framework (MRF). It was originally proposed in [1,2], and we recently studied it as an approximation method for piecewise smooth functions [3].

Harten's Multiresolution Framework

Let us consider $f : [0,1] \longrightarrow \mathbb{R}$. Harten's MRF is based on a multiscale data representation, which relies in a set of nested grids that lead to two basic operations:

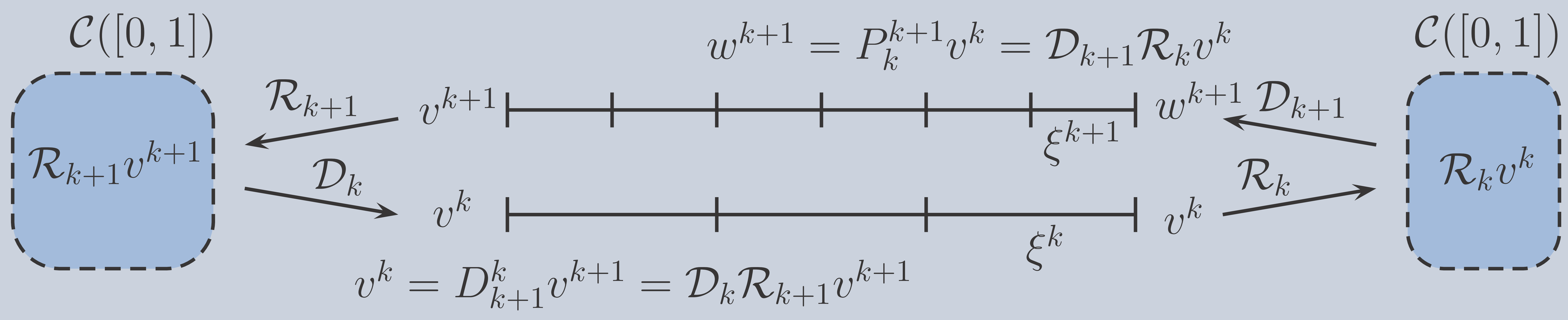
- $\mathcal{D}_k$: Discretization operator. Example: $\mathcal{D}_k f = \left(f(\xi_i^k)\right)_i =: v^k, \quad \xi_i^k = i2^{-k}$.
- $\mathcal{R}_k$: Reconstruction operator. Examples: Polynomial or ENO interpolation.



Compatibility condition: $\mathcal{D}_k\mathcal{R}_k v^k = v^k$.

Handling data between grids:

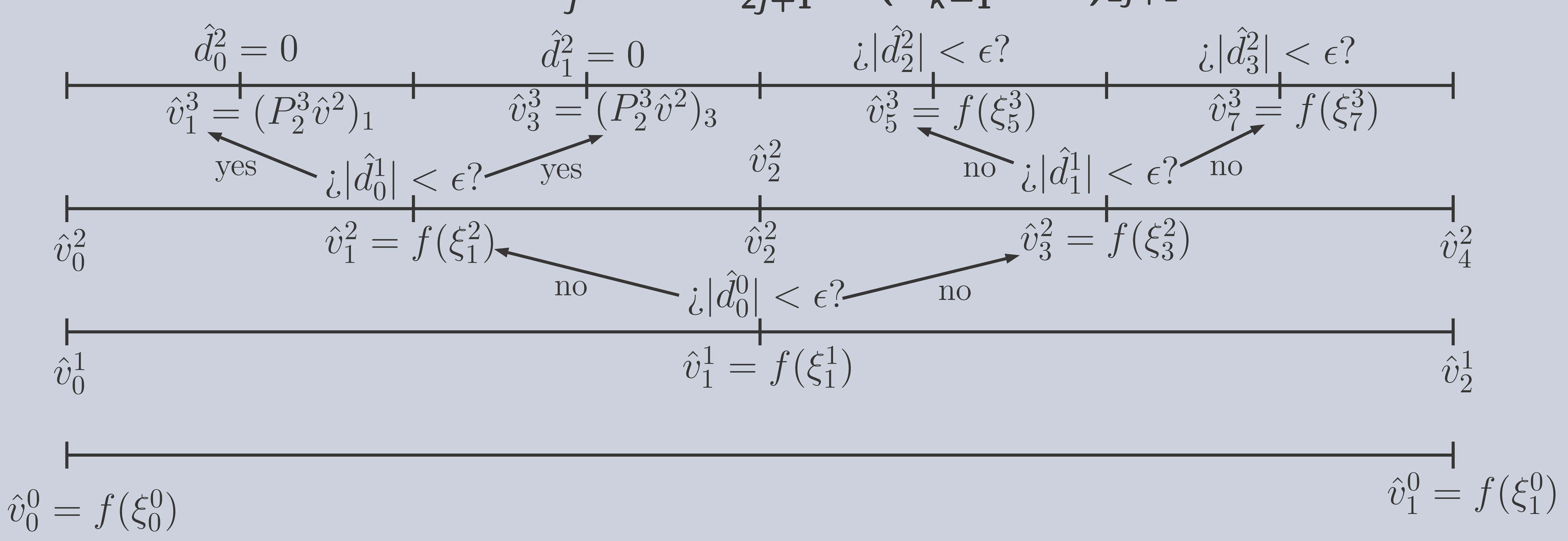
- $D_{k+1}^k = \mathcal{D}_k\mathcal{R}_{k+1}$: Decimation operator. Example: $D_{k+1}^k v^{k+1} = \left(v_{2i}^{k+1}\right)_{i=0}^{2^K}$
- $P_k^{k+1} = \mathcal{D}_{k+1}\mathcal{R}_k$: Prediction or **subdivision** operator. Example: PCHIP rule,
$$(P_k^{k+1}v^k)_{2i+1} = \frac{1}{2}v_i^k + \frac{1}{2}v_{i+1}^k - \frac{1}{8}\left(H(\nabla v_i^k, \nabla v_{i+1}^k) - H\nabla(v_{i-1}^k, \nabla v_i^k)\right),$$
H is the harmonic mean.
- Detail coefficient: $d_i^k := v_{2i+1}^{k+1} - (P_k^{k+1}v^k)_{2i+1}$



Approximation order s: $d_i^k = O(2^{-ks}), \quad \forall i, k$

Truncate and Encode - Adaptive approximation strategy

Goal: Given $\epsilon > 0$, obtain some $\hat{v}^K \in \mathbb{R}^{2^K+1}$ such as $\|v^K - \hat{v}^K\|_\infty \lessapprox \epsilon$, but using as few evaluations of f as we can. Why? Because f could be hard to evaluate.
Start with $\hat{v}^0 = \mathcal{D}_0 f$, $\hat{v}^1 = \mathcal{D}_1 f$ and compute recursively:
$$\hat{v}_{2i+1}^{k+1} = \begin{cases}(P_k^{k+1}\hat{v}^k)_{2i+1}, & \text{if } |\hat{d}_j^{k-1}| < \epsilon\\ f(\xi_{2i+1}^{k+1}), & \text{if } |\hat{d}_j^{k-1}| \geq \epsilon\end{cases}, \quad \forall i \in \{2j, 2j+1\}$$
Modified detail coefficient: $\hat{d}_j^{k-1} := \hat{v}_{2j+1}^k - (P_{k-1}^k\hat{v}^{k-1})_{2j+1}$.



References

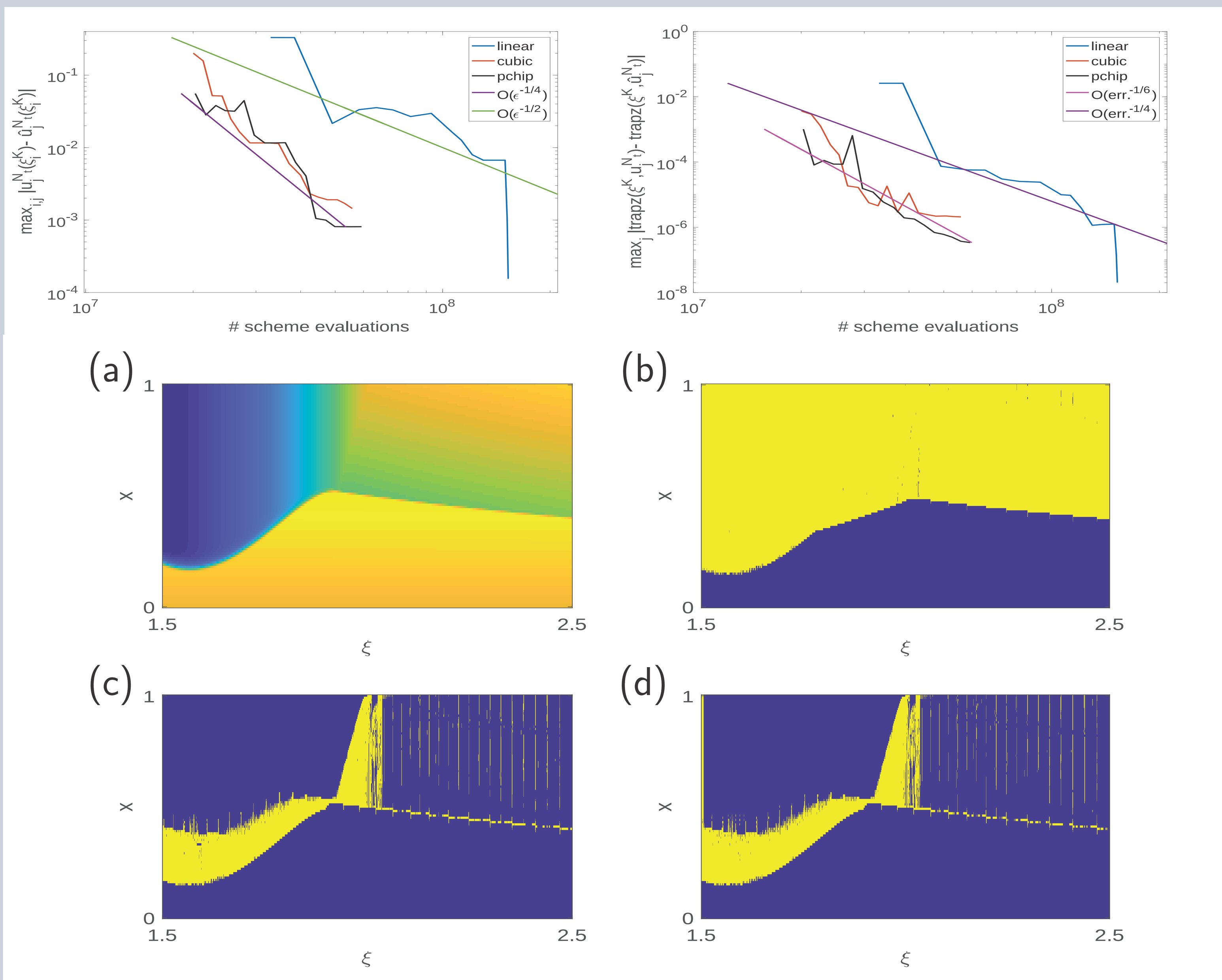
[1] R. Abgrall, P.M. Congedo, and G. Geraci. *A one-time truncate and encode multiresolution stochastic framework*. J. Comput. Phys. 257, 19-56 (2014)
[2] G. Geraci, P.M. Congedo, R. Abgrall, and G. Laccarino. *A novel weakly-intrusive non-linear multiresolution framework for uncertainty quantification in hyperbolic partial differential equations*. J. Sci. Comput. 66, 358-405 (2016)
[3] S. López-Ureña and R. Donat. *High-accuracy approximation of piecewise smooth functions using the truncation and encode approach*. Applied Mathematics and nonlinear sciences, 2, 367-384 (2017)

Truncate and Encode - PDE in Uncertainty Quantification

Solve a PDE with stochastic data:
$$\partial_t u(x,t,\xi) + \partial_x\phi(x,t,\xi,u(x,t,\xi)) = 0,$$
where ξ is a random variable. For example, $\xi \sim \mathcal{U}[0,1]$.
Goal: Compute u or some related statistic, like the expectancy
$$E(x,t) = \int_0^1 u(x,t,\xi)p(\xi)dx \approx \sum_i \alpha_i u(x,t,\xi_i^K)p(\xi_i^K),$$
on a space-time grid of $N_x \cdot N_t$ points. For this, take a numerical scheme to compute $u_j^n(\xi) \approx u(x_j, t^n, \xi)$, and denote it as a function
$$u_j^n(\xi) =: f_j^{n-1}(\xi), \qquad j = 0,1,\ldots,N_x, \quad n = 1,2,\ldots,N_t.$$
The numerical schemes may be hard to evaluate. **Idea:** Apply Truncate and Encode to approximate $\left(f_j^n(\xi_i^K)\right)_i$, for each j, n.

Numerical experiments

We solved the Burguers equation, $\phi(x,t,u) = u^2/2$, for the initial data $u_0(x,\xi) = \sin(\pi x\xi)$, with parameters $\xi \in [1.5, 2.5]$, $x \in [0,1]$, $N_x = 100$, CFL $= 0.8$, $\xi^k = (i2^{-k}/3)_{i=0}^{2^k3}$, $k = 0,1,\ldots,11$. We used the second order MHM scheme [2]. We computed the expectancy with $p(\xi) = 1$ using the trapezoidal rule. The number of times that the scheme must be applied to compute the solution (u^{N_t}) without TE is 155,781,895.



Top figures: For the linear, cubic and PCHIP interpolations, the numb. times the scheme $f_j^n(\xi)$ is used vs. the error $\max_{i,j}|u_j^{N_t}(\xi_i^K) - \hat{u}_j^{N_t}(\xi_i^K)|$ (left) and $\max_j|trapz(\xi^K,(u_j^{N_t}(\xi_i^K))_{i=0}^{2^k3}) - trapz(\xi^K,(u_j^{N_t}(\xi_i^K))_{i=0}^{2^k3})|$ (right). Figure (a): The approximated solution without TE u^{N_t}. Figures (b), (c), (d): The linear, cubic and PCHIP interp. (respectively) were used (blue (ξ, x) points) instead of the scheme (yellow points). The corresponding errors are: (b) 6.6990e-3, (c) 1.9057e-3 and (d) 8.1156e-4.

Conclusions

- The TE algorithm drastically reduces the computation time when simulating PDEs with stochastic parameters.
- It can be safely applied even in presence of discontinuities.
- The efficiency (the ration between error and computation time) strongly depends on the approximation capabilities of the prediction operator.

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