Variational methods for non-variational problems

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Variational problems

For $\mathbf{x}(t) : [0,1] \to \mathbb{R}^N$ belonging to a suitable class of paths,

$$E(\mathbf{x}) = \int_0^1 \phi(\mathbf{x}(t), \mathbf{x}'(t))\, dt, \quad \phi(\mathbf{y}, \mathbf{z}) : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R},$$

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Optimality system

$$-[\phi_{\mathbf{z}}(\mathbf{x}(t), \mathbf{x}'(t))]' + \phi_{\mathbf{x}}(\mathbf{x}(t), \mathbf{x}'(t)) = \mathbf{0} \text{ in } (0,1),$$

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Comparison

$$-[\mathbf{f}(\mathbf{x}(t), \mathbf{x}'(t)]' + \mathbf{g}(\mathbf{x}(t), \mathbf{x}'(t)) = \mathbf{0} \text{ in } (0, 1), \quad \mathbf{x}(0) = \mathbf{x}(1) = \mathbf{0}.$$

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1. the infimum $m \geq 0$ vanishes, and
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- E is not a typical integral functional; hard to apply weak lower semicontinuity based on convexity.
- Have to rely on smoothness and compactness through the classical Palais-Smale condition.

Assumptions, and main result

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1. Linear growth in derivatives: $f_1(\mathbf{x})$, $f_0(\mathbf{x})$, $g_1(\mathbf{x})$, $g_0(\mathbf{x})$, and $C_0, C_2 \geq 0$ such that

$$|\mathbf{f}(\mathbf{x}, \mathbf{z})| \leq f_1(\mathbf{x})|\mathbf{z}| + f_0(\mathbf{x}), \quad |\mathbf{g}(\mathbf{x}, \mathbf{z})| \leq g_1(\mathbf{x})|\mathbf{z}| + g_0(\mathbf{x}),$$
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$$C_1|\mathbf{y}|^2 \leq \mathbf{y}\mathbf{f}_{\mathbf{z}}(\mathbf{x},\mathbf{z}) \cdot \mathbf{y}$$
$$\|\mathbf{f}_{\mathbf{x}}\|_{L^\infty(\mathbb{R}^N \times \mathbb{R}^N)} + \|\mathbf{g}_{\mathbf{z}}\|_{L^\infty(\mathbb{R}^N \times \mathbb{R}^N)} + \|\mathbf{g}_{\mathbf{x}}\|_{L^\infty(\mathbb{R}^N \times \mathbb{R}^N)} < C_1.$$

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Theorem

Under indicated assumptions on the maps $\mathbf{f}$ *and* $\mathbf{g}$*, there is at least one solution* $\mathbf{x}$*.*

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Proposition

$E : \mathbb{H} \to \mathbb{R}$, a smooth functional over a Hilbert space $\mathbb{H}$. E is coercive ($E(\mathbf{x}) \to \infty$ if $\|\mathbf{x}\| \to \infty$), and enjoys the Palais-Smale property. Then there are minimizers for E in $\mathbb{H}$.

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Palais-Smale condition.
Every sequence $\{\mathbf{x}_j\}$ such that $\{E(\mathbf{x}_j)\}$ is a bounded sequence of numbers, and $E'(\mathbf{x}_j) \to \mathbf{0}$ in $\mathbb{H}$, admits a subsequence converging (strongly) in that same space.

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But also need: such minimizer $\mathbf{x}$ of E in $\mathbb{H}$ is such that $E(\mathbf{x}) = 0$.

A non-local, integral form

Operator:

$$\mathbf{x} \mapsto \mathbf{y} : \quad -[\mathbf{f}(\mathbf{x}, \mathbf{x}') + \mathbf{y}']' + \mathbf{g}(\mathbf{x}, \mathbf{x}') = \mathbf{0} \text{ in } (0,1), \quad \mathbf{y}(0) = \mathbf{y}(1) = \mathbf{0}.$$

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$$\begin{aligned}
\mathbf{y}(t) = &(t-1)\int_0^1 [\mathbf{f}(\mathbf{x}(s), \mathbf{x}'(s)) - (1-s)\mathbf{g}(\mathbf{x}(s), \mathbf{x}'(s))]\, ds \\
&+ (t-1)\int_0^t \mathbf{g}(\mathbf{x}(s), \mathbf{x}'(s))\, ds \\
&+ \int_t^1 [\mathbf{f}(\mathbf{x}(s), \mathbf{x}'(s)) - (1-s)\mathbf{g}(\mathbf{x}(s), \mathbf{x}'(s))]\, ds,
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$$\begin{aligned}\mathbf{y}(t) =&(t-1)\int_0^1 [\mathbf{f}(\mathbf{x}(s), \mathbf{x}'(s)) - (1-s)\mathbf{g}(\mathbf{x}(s), \mathbf{x}'(s))]\,ds \\ &+ (t-1)\int_0^t \mathbf{g}(\mathbf{x}(s), \mathbf{x}'(s))\,ds \\ &+ \int_t^1 [\mathbf{f}(\mathbf{x}(s), \mathbf{x}'(s)) - (1-s)\mathbf{g}(\mathbf{x}(s), \mathbf{x}'(s))]\,ds,\end{aligned}$$

$$\begin{aligned}E(\mathbf{x}) = \frac{1}{2}\int_0^1 \Big| &\int_0^1 [\mathbf{f}(\mathbf{x}(s), \mathbf{x}'(s)) - (1-s)\mathbf{g}(\mathbf{x}(s), \mathbf{x}'(s))]\,ds \\ &+ \int_0^t \mathbf{g}(\mathbf{x}(s), \mathbf{x}'(s))\,ds - \mathbf{f}(\mathbf{x}(t), \mathbf{x}'(t))\Big|^2\,dt.\end{aligned}$$

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$$\langle E'(\mathbf{x}), \mathbf{X}\rangle = \int_0^1 \mathbf{y}'(t) \cdot \mathbf{Y}'(t)\, dt$$

$$= -\int_0^1 [\mathbf{y}' \cdot (\mathbf{f}_{\mathbf{x}}(\mathbf{x}, \mathbf{x}')\mathbf{X} + \mathbf{f}_{\mathbf{z}}(\mathbf{x}, \mathbf{x}')\mathbf{X}') + \mathbf{y} \cdot (\mathbf{g}_{\mathbf{x}}(\mathbf{x}, \mathbf{x}')\mathbf{X} + \mathbf{g}_{\mathbf{z}}(\mathbf{x}, \mathbf{x}')\mathbf{X}')]\, dt$$

$$= -\int_0^1 [(\mathbf{y}'\mathbf{f}_{\mathbf{x}}(\mathbf{x}, \mathbf{x}') + \mathbf{y}\mathbf{g}_{\mathbf{x}}(\mathbf{x}, \mathbf{x}')) \cdot \mathbf{X} + (\mathbf{y}'\mathbf{f}_{\mathbf{z}}(\mathbf{x}, \mathbf{x}') + \mathbf{y}\mathbf{g}_{\mathbf{z}}(\mathbf{x}, \mathbf{x}')) \cdot \mathbf{X}']\, dt.$$

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$$E'(\mathbf{x}) = \mathbb{X} : \quad -[\mathbb{X}' + \mathbf{y}'\mathbf{f}_\mathbf{z}(\mathbf{x}, \mathbf{x}') + \mathbf{y}\mathbf{g}_\mathbf{z}(\mathbf{x}, \mathbf{x}')]' + \mathbf{y}'\mathbf{f}_\mathbf{x}(\mathbf{x}, \mathbf{x}') + \mathbf{y}\mathbf{g}_\mathbf{x}(\mathbf{x}, \mathbf{x}') = \mathbf{0}.$$

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Coercivity: $\mathbf{x}$ itself as a test function

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Assumptions lead to:

$$C_1\|\mathbf{x}'\|^2_{L^2(0,1;\mathbb{R}^N)} - C_0\|\mathbf{x}\|_{L^\infty(0,1;\mathbb{R}^N)} \le \|\mathbf{y}'\|_{L^2(0,1;\mathbb{R}^N)}\|\mathbf{x}'\|_{L^2(0,1;\mathbb{R}^N)} + C\|\mathbf{x}'\|_{L^2(0,1;\mathbb{R}^N)}.$$

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Palais-Smale condition

$\{\mathbf{x}_j\}$, $\{E(\mathbf{x}_j)\}$, bounded and $E'(\mathbf{x}_j) \to \mathbf{0}$ in $H_0^1(0,1;\mathbb{R}^N)$: admits a converging subsequence (strongly).

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$\mathbb{X}_j = E'(\mathbf{x}_j)$:

$$-[\mathbb{X}_j' + \mathbf{y}_j'\mathbf{f}_\mathbf{z}(\mathbf{x}_j, \mathbf{x}_j') + \mathbf{y}_j\mathbf{g}_\mathbf{z}(\mathbf{x}_j, \mathbf{x}_j')]' + \mathbf{y}_j'\mathbf{f}_\mathbf{x}(\mathbf{x}_j, \mathbf{x}_j') + \mathbf{y}_j\mathbf{g}_\mathbf{x}(\mathbf{x}_j, \mathbf{x}_j') = \mathbf{0}.$$

Palais-Smale condition

$\{\mathbf{x}_j\}$, $\{E(\mathbf{x}_j)\}$, bounded and $E'(\mathbf{x}_j) \to \mathbf{0}$ in $H_0^1(0,1;\mathbb{R}^N)$: admits a converging subsequence (strongly).

Coercivity: $\mathbf{x}_j' \rightharpoonup \mathbf{x}'$, $\mathbf{x}_j \to \mathbf{x}$.
Objective: $\|\mathbf{x}_j' - \mathbf{x}'\|_{L^2(0,1;\mathbb{R}^N)} \to 0$.

$\mathbb{X}_j = E'(\mathbf{x}_j)$:

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Also know:

$$-[\mathbf{f}(\mathbf{x}_j, \mathbf{x}_j') + \mathbf{y}_j']' + \mathbf{g}(\mathbf{x}_j, \mathbf{x}_j') = \mathbf{0}, \quad -[\mathbf{f}(\mathbf{x}, \mathbf{x}') + \mathbf{y}']' + \mathbf{g}(\mathbf{x}, \mathbf{x}') = \mathbf{0}.$$

Two steps

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1. $E(\mathbf{x}_j) = (1/2)\|\mathbf{y}_j'\|^2 \to 0$: $\mathbf{y}_j$, test function

$$\int_0^1 [(\mathbb{X}_j' + \mathbf{y}_j'\mathbf{f}_\mathbf{z}(\mathbf{x}_j, \mathbf{x}_j') + \mathbf{y}_j\mathbf{g}_\mathbf{z}(\mathbf{x}_j, \mathbf{x}_j')) \cdot \mathbf{y}_j' \\ + (\mathbf{y}_j'\mathbf{f}_\mathbf{x}(\mathbf{x}_j, \mathbf{x}_j') + \mathbf{y}_j\mathbf{g}_\mathbf{x}(\mathbf{x}_j, \mathbf{x}_j')) \cdot \mathbf{y}_j]\, dt = 0.$$

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$$\begin{aligned} C_1\|\mathbf{y}_j'\|^2 \le &- \int_0^1 [(\mathbb{X}_j' + \mathbf{y}_j(\mathbf{g}_\mathbf{z}(\mathbf{x}_j, \mathbf{x}_j') + \mathbf{f}_\mathbf{x}(\mathbf{x}_j, \mathbf{x}_j')^T) \cdot \mathbf{y}_j' \\ &+ \mathbf{y}_j\mathbf{g}_\mathbf{x}(\mathbf{x}_j, \mathbf{x}_j') \cdot \mathbf{y}_j]\, dt \\ \le &\|\mathbb{X}_j'\|\,\|\mathbf{y}_j'\| + \Big(\|\mathbf{f}_\mathbf{x}\|_{L^\infty(\mathbb{R}^N\times\mathbb{R}^N)} + \|\mathbf{g}_\mathbf{z}\|_{L^\infty(\mathbb{R}^N\times\mathbb{R}^N)} \\ &+ \|\mathbf{g}_\mathbf{x}\|_{L^\infty(\mathbb{R}^N\times\mathbb{R}^N)}\Big)\,\|\mathbf{y}_j'\|^2_{L^2(0,1;\mathbb{R}^N)}. \end{aligned}$$

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2. $\|\mathbf{x}_j' - \mathbf{x}'\| \to 0$: $\mathbf{x}_j - \mathbf{x}$, test function

$$\int_0^1 [(\mathbf{f}(\mathbf{x}_j, \mathbf{x}_j') - \mathbf{f}(\mathbf{x}, \mathbf{x}') + (\mathbf{y}_j' - \mathbf{y}')) \cdot (\mathbf{x}_j' - \mathbf{x}') + (\mathbf{g}(\mathbf{x}_j, \mathbf{x}_j') - \mathbf{g}(\mathbf{x}, \mathbf{x}')) \cdot (\mathbf{x}_j - \mathbf{x})]\, dt$$

Conclusion

$$C_1\|\mathbf{x}_j'-\mathbf{x}'\|^2 \le \|\mathbf{y}_j'\|\,\|\mathbf{x}_j'-\mathbf{x}'\|+\int_0^1 [\mathbf{y}'\cdot(\mathbf{x}_j'-\mathbf{x}')+(\mathbf{g}(\mathbf{x}_j,\mathbf{x}_j')-\mathbf{g}(\mathbf{x},\mathbf{x}'))(\mathbf{x}_j-\mathbf{x})]\,dt.$$

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Interesting consequence

Under our hypotheses on the mappings **f** and **g**, we have

$$\lim_{E'(\mathbf{x})\to 0} E(\mathbf{x}) = 0.$$

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Proof of main result.

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Proof of main result.

- $\{\mathbf{x}_j\}$, minimizing for E: $E'(\mathbf{x}_j) \to 0$, and $\mathbf{x}_j' \rightharpoonup \mathbf{x}$;
- $E(\mathbf{x}_j) \to 0$, $\mathbf{x}_j \to \mathbf{x}$ (strong).
- $\mathbf{x}$, a solution: $E(\mathbf{x}) = 0$.

Fully non-linear problems in non-divergence form

$$\mathbf{F}(\mathbf{x}(t), \mathbf{x}'(t), \mathbf{x}''(t)) = \mathbf{0} \text{ in } (0,1), \quad \mathbf{x}(0) = \mathbf{x}(1) = \mathbf{0}.$$

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Theorem

Under appropriate assumptions, there is at least one solution $\mathbf{x}$.

Assumptions (for a scalar case)

1. Smoothness: F is smooth with respect to the variables (x, p, q);
2. Convexity: the function $|F(t, x, p, q)|^2$ is a convex function of q;
3. Coercivity: there are positive constants C and M with
$$|F(t,x,p,q)| \geq C|q| - M(|p| + |x| + 1), \quad |x'(0)| \leq C(E(x) + 1);$$
4. Growth: there is a locally bounded function $C(x, p)$ such that
$$|F(t,x,p,q)F_q(t,x,p,q)| \leq C(x,p)(1 + |q|),$$
$$|F(t,x,p,q)F_p(t,x,p,q)| \leq C(x,p)(1 + |q|^2),$$
$$|F(t,x,p,q)F_x(t,x,p,q)| \leq C(x,p)(1 + |q|^2);$$
5. Positivity: we always have
$$F_q > 0, \quad F_x F_q + \frac{1}{4}|F_p|^2 \leq 0,$$
and if x is such that
$$F_x F_q + \frac{1}{4}|F_p|^2 \equiv 0 \text{ in } (0,1),$$
then $F \equiv 0$ in $(0,1)$ as well.