Regularity of solutions of elliptic problems with Dirac measures as data

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Motivation

In this talk we mainly study the problem

$$\begin{cases} -\Delta v = q\delta_\sigma, & \text{in } Q, \\ v = 0, & \text{on } \partial Q, \end{cases} \tag{1}$$

where Q is a subset of $\mathbb{R}^3$, σ is a one-dimensional curve strictly included in Q and q belongs to $L^2(\sigma)$.

Such problems occur:

1. in reduced models of fluid flows (Darcy's law in fractured domains): in order to save computational resources, see [Barenblatt-Entov-Ryzhick 90, Fabrie-Gallouët 00, D'Angelo-Quarteroni 08],

2. in physical problems: corresponds to the idealization of a load supported by σ.

First difficulty: $v \notin H^1_0(Q)$ because the Dirac mass $\notin H^{-1}(Q)$.

Main Goal: Regularity results (useful for numerical applications).

Main results

We give such regularity results for two model problems:
1. σ is a full line,
2. σ a half-line.

For 1. we use Fourier transform technique, while for 2. we use a Mellin transformation. In both cases we are reduced to a Helmholtz problem in dimension two.
For this last problem we prove uniform estimates in standard or weighted Sobolev spaces, taking the transformation back we obtain the expected regularity result.

Plan

Weighted Sobolev spaces

Let α be an arbitrary real number, Ω be a domain of $\mathbb{R}^n$ and r be the distance to a part P of $\overline{\Omega}$ (that could vary at different occurences).

- $L^2_\alpha(\Omega; P)$ the Hilbert space made of measurable functions u st

$$\|u\|^2_{L^2_\alpha(\Omega;P)} = \int_\Omega |u(x)|^2 r^{2\alpha}(x)dx < \infty.$$

- For $m \in \mathbb{N}$, we define the weighted Sobolev space

$$H^m_\alpha(\Omega; P) = \{u \in L^2_\alpha(\Omega; P) \mid D^\gamma u \in L^2_\alpha(\Omega; P), \forall |\gamma| \leq m\}.$$

- The weighted Sobolev space of Kondratiev's type is defined by

$$V^m_\alpha(\Omega; P) = \{u \in L^2_{\alpha-m}(\Omega; P) \mid r^{\alpha-m+|\gamma|} D^\gamma u \in L^2(\Omega), \forall |\gamma| \leq m\}.$$

Proposition 1.1

Let Ω be a bounded domain of $\mathbb{R}^n$ s.t. $0 \in \Omega$ and let $m \in \mathbb{N}^*$. For all $\alpha > m - \frac{n}{2}$ we have

$$H^m_\alpha(\Omega; 0) = V^m_\alpha(\Omega; 0).$$

Pf Based on Hardy's inequalities.

Regularity in weighted Sobolev spaces

Theorem 1.3

Let Ω be a bd domain of $\mathbb{R}^2$ with a C^m bdy, $m \geq 2$ or a C^m compact manifold without bdy. Fix a point $B \in \Omega$. Let L be an elliptic op. of order 2 with coeff. in $C^m(\bar{\Omega})$ and $\alpha \in \mathbb{R}$. Let $u \in H^2_{loc}(\Omega \setminus \{B\})$ be a solution of

$$\begin{cases} Lu = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

such that $u \in V^1_{\alpha-m+1}(\Omega; B)$ and $f \in V^{m-2}_{\alpha}(\Omega; B)$. Then $u \in V^m_{\alpha}(\Omega; B)$ with the estimate

$$\|u\|_{V^m_\alpha(\Omega;B)} \lesssim \|f\|_{V^{m-2}_\alpha(\Omega;B)} + \|u\|_{V^1_{\alpha-m+1}(\Omega;B)}.$$

Pf Based on a localization argument and a diadic covering.

Proposition 1.2

Let Ω be a bounded domain of $\mathbb{R}^n$ with a lip. bdy and let $m \in \mathbb{N}^*$. $\forall \lambda \in [0, \infty[$, $w \in H^m(\Omega)$, set

$$\|w\|^2_{m,\Omega,\lambda} := \sum_{l=0}^{m} \lambda^{2l} \|w\|^2_{m-l,\Omega}.$$

Then $$\|w\|^2_{m,\Omega,\lambda} \sim (1 + \lambda^{2m}) \|w\|^2_{0,\Omega} + \|w\|^2_{m,\Omega}.$$

Pf Based on interpolation inequalities.

Plan

Fundamental solution

Definition 1.4

For $k \in \mathbb{C} \setminus \{0\}$, we define

$$
\begin{aligned}
H_k(x,y) &:= \frac{i}{4} H_0^{(1)}(k\sqrt{x^2+y^2}), \forall (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}, \\
H_0(x,y) &:= \frac{1}{2\pi} \ln \sqrt{x^2+y^2}, \forall (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\},
\end{aligned}
$$

where $H_0^{(1)}(z)$ is the Hankel function of type 1 and order 0.

Fund. sol.: $(\Delta + k^2) H_k = \delta_0$ in $\mathbb{R}^2$.

Regularity of H_k

Theorem 1.5

Let Ω be a bounded domain of $\mathbb{R}^2$, $m \geq 0$ and $\alpha > m-1$. Let $k \in \mathbb{C} \backslash \{0\}$ s. t. $\frac{\pi}{4} \leq \arg k \leq \frac{3\pi}{4}$. Then

$$\mathcal{H}_k = \begin{cases} H_k & \text{if } |k| > 1, \\ H_0 & \text{if } |k| \leq 1, \end{cases}$$

belongs to $V_\alpha^m(\Omega; 0)$ with

$$\sum_{l=0}^{m} |k|^{2l} \|\mathcal{H}_k\|^2_{V_\alpha^{m-l}(\Omega;0)} \lesssim 1.$$

Pf Behavior of H_k and H_0 near 0 and ∞.

Plan

Reg. of the sol. in Sobolev spaces with a parameter

Theorem 3.1

Let $\Omega \subset \mathbb{R}^2$ be bounded, let $h \in L^2(\Omega)$ and $k \in \mathbb{C}$ s. t. $\frac{\pi}{4} \leq \arg k \leq \frac{3\pi}{4}$. Let $w \in H^1_0(\Omega)$ sol. of

$$\left\{ \begin{array}{rl} (\Delta + k^2)w = h, & \text{in } \Omega, \\ w = 0, & \text{on } \partial\Omega. \end{array} \right. \tag{2}$$

1. Then we have

$$\|w\|_{1,\Omega,|k|+1} \lesssim \|h\|_{0,\Omega}.$$

2. Moreover, $\forall m \geq 2$, if $\partial\Omega \in \mathcal{C}^m$ and $h \in H^{m-2}(\Omega)$ then $w \in H^m(\Omega)$ with

$$\|w\|_{m,\Omega,|k|+1} \lesssim \|h\|_{m-2,\Omega,|k|+1}. \tag{3}$$

Proof

For point 1, we use the variational formulation

$$\forall v \in H_0^1(\Omega), \quad \int_\Omega \nabla w \, \overline{\nabla v} - k^2 \int_\Omega w \, \overline{v} = - \int_\Omega h \, \overline{v}.$$

and Poincaré's inequality.
For point 2, we use a priori estimates of ADN and a bootstrap argument.

Plan

Helmholtz pb with a Dirac mass

Let Ω be a bounded domain of $\mathbb{R}^2$ s. t. $0 \in \Omega$ and let $k \in \mathbb{C}$ s. t. $\frac{\pi}{4} \leq \arg k \leq \frac{3\pi}{4}$.
Consider the pb

$$\begin{cases} (\Delta + k^2)u_k = \delta_0 & \text{in } \Omega, \\ u_k = 0 & \text{on } \partial\Omega. \end{cases}$$

The case $k = 0$ was treated in [Fabrie-Gallouët 00, Araya-Behrens-Rodriguez 06], where they prove the existence of a sol. in $W^{1,p}(\Omega)$ for all $p < 2$.
GOALS:
1. $u_k \in W^{1,p}(\Omega)$ for all $p < 2$ (weak sol.)
2. $u_k \in V^m_\alpha(\Omega; 0)$, for all $m \geq 1$, $\alpha > m - 1$.

Tools

Use the fund. sol.

$$\mathcal{H}_k := \begin{cases} H_k & \text{if } |k| > 1, \\ H_0 & \text{if } |k| \leq 1. \end{cases}$$

Fix η a cut-off fct. s. t. for $\delta \in]0, \frac{dist(0,\partial\Omega)}{2}[$ fixed:

$$\begin{cases} \eta = 1 & \text{on } B(0,\delta), \\ \eta = 0 & \text{on } B(0,2\delta)^c, \end{cases}$$

and decompose u_k in the form

$$u_k = \eta \mathcal{H}_k + w_k, \tag{4}$$

Existence and uniqueness of a weak sol.

where w_k is sol. of

$$\begin{cases} (\Delta + k^2) w_k = h_k & \text{in } \Omega, \\ w_k = 0 & \text{on } \partial\Omega, \end{cases}$$

$$h_k = \begin{cases} 2\nabla\eta\nabla H_k + \Delta\eta H_k & \text{if } |k| > 1, \\ \\ 2\nabla\eta\nabla H_0 + \Delta\eta H_0 - k^2\eta H_0 & \text{if } |k| \leq 1. \end{cases}$$

- $\mathcal{H}_k \in W^{1,p}(\Omega)$ for all $p < 2$.
- $h_k \in L^2(\Omega) \Rightarrow$ var. form. + Lax Milgram $\Rightarrow w_k \in H_0^1(\Omega)$.

Higher regularity of the solution

Theorem 3.2

Let $m \geq 1$, $\alpha > m-1$ and Ω be a bounded domain of $\mathbb{R}^2$ of class $\mathcal{C}^m$ s. t. $0 \in \Omega$. Let $k \in \mathbb{C}$ s. t. $\frac{\pi}{4} \leq \arg k \leq \frac{3\pi}{4}$. Then the sol. u_k of

$$\begin{cases} (\Delta + k^2)u_k = \delta_0 & \text{in } \Omega, \\ u_k = 0 & \text{on } \partial\Omega, \end{cases} \tag{5}$$

belongs to $V_\alpha^m(\Omega; 0)$ with

$$\sum_{l=0}^{m} |k|^{2l} \|u_k\|^2_{V_\alpha^{m-l}(\Omega;0)} \lesssim 1.$$

Proof

According to the splitting (4), it suffices to estimate $\mathcal{H}_k$ and w_k.
By Thm. 1.5, we have $\mathcal{H}_k \in V^m_\alpha(\Omega; 0)$ for $\alpha > m - 1$ and the right estimate.
First case: $|k| > 1$. As $H^{m-l}(\Omega) \hookrightarrow V^{m-l}_\alpha(\Omega; 0)$, it suffices to show that

$$\|w_k\|^2_{m,\Omega,|k|} = \sum_{l=0}^{m} |k|^{2l} \|w_k\|^2_{m-l,\Omega} \lesssim 1.$$

The remainder of the proof is based on the application of Thm. 3.1 by using the reg. of H_k in weighted Sobolev spaces.

Proof ctd

Second case: $|k| \leq 1$. We only need to check that

$$\|w_k\|^2_{V_\alpha^m(\Omega;0)} \lesssim 1, \tag{6}$$

holds. For $m = 1$, we notice that

$$\|h_k\|_{0,\Omega} \lesssim |H_0|_{V_\epsilon^1(\Omega;0)} + \|H_0\|_{V_{\epsilon-1}^0(\Omega;0)} \lesssim 1$$

for any $0 < \epsilon < 1$. Hence by Thm. 3.1 we deduce that

$$\|w_k\|_{1,\Omega} \lesssim 1,$$

that shows (6) for $m = 1$ since for $\alpha > 0$,
$H^1(\Omega) \hookrightarrow H_\alpha^1(\Omega;0) \hookrightarrow V_\alpha^1(\Omega;0)$ due to Prop. 1.1
For $m \geq 2$, we use an iterative argument by looking at w_k as solution of

$$\Delta w_k = h_k - k^2 w_k.$$

Then we apply Thm. 1.3.

The Laplace eq. in 3D with an infinite fracture

Theorem 2.1

Let $m \in \mathbb{N}^*$, $\alpha > m - 1$ and $Q = \Omega \times \mathbb{R}$ where Ω is a bounded domain of $\mathbb{R}^2$, of class $\mathcal{C}^m$ s. t. $0 \in \Omega$. Let σ be the z-axis, that is the infinite fracture. Let $q \in L^2(\mathbb{R})$. Then the solution u of

$$\begin{cases} -\Delta u = q\delta_\sigma, & \text{in } Q, \\ u = 0, & \text{on } \partial Q, \end{cases} \tag{7}$$

belongs to $V^m_\alpha(Q;\sigma)$ with the estimate

$$\|u\|_{V^m_\alpha(Q;\sigma)} \lesssim \|q\|_{0,\mathbb{R}}.$$

Pf

Applying partial Fourier transform in z to (7), we observe that $\mathfrak{F}_z(u)$ is sol. of

$$\begin{cases} (-\Delta + \xi^2)\mathfrak{F}_z(u) = \mathfrak{F}(q)\delta_0 & \text{in } \Omega, \\ \mathfrak{F}_z(u) = 0 & \text{on } \partial\Omega, \end{cases}$$

Hence $\mathfrak{F}_z(u(x,y,\cdot))(\xi) = \mathfrak{F}(q)(\xi)\, u_\xi(x,y)$ where u_ξ is sol. of

$$\begin{cases} (-\Delta + \xi^2)u_\xi = \delta_0 & \text{in } \Omega, \\ u_\xi = 0 & \text{on } \partial\Omega, \end{cases}$$

By Thm. 3.2, we get

$$\sum_{l=0}^{m} |\xi|^{2l} \|u_\xi\|^2_{V_\alpha^{m-l}(\Omega;0)} \lesssim 1. \tag{8}$$

Pf

On the other hand Parseval's identity implies that

$$\|u\|^2_{V^m_\alpha(Q;\sigma)} = \frac{1}{2\pi}\sum_{\beta_3=0}^{m}\int_{\mathbb{R}} |\xi|^{2\beta_3}\|\mathfrak{F}_z(u)\|^2_{V^{m-\beta_3}_\alpha(\Omega;0)}d\xi. \qquad (9)$$

By (8) and (9), we obtain

$$\|u\|^2_{V^m_\alpha(Q;\sigma)} \lesssim \int_{\mathbb{R}} |\mathfrak{F}(q)(\xi)|^2 d\xi.$$

Again Parseval $\Rightarrow$

$$\|u\|_{V^m_\alpha(Q;\sigma)} \lesssim \|q\|_{0,\mathbb{R}}.$$

The Laplace eq. in 3D with a semi infinite fracture

Consider the pb

$$-\Delta u = q\,\delta_\sigma \quad \text{in } \mathbb{R}^3, \tag{10}$$

where σ is the semi-axis of positive x.

Theorem 4.1

Let $m \geq 1$ and $m-1 < \alpha < m - \frac{1}{2}$. Let $q \in L^2_\epsilon(\mathbb{R}^+)$ with $\epsilon = \alpha - (m-1)$. Then the sol. u of pb (10) belongs to $V^m_\alpha(\mathbb{R}^3;\sigma)$ with

$$\|u\|_{V^m_\alpha(\mathbb{R}^3;\sigma)} \lesssim \|q\|_{L^2_\epsilon(\mathbb{R}^+)}.$$

Pf Mellin transformation and use of Thm. 1.3.

The Mellin transform

For a function $u \in \mathcal{D}(0, \infty)$ the Mellin transform $M[u]$ of u is defined at $\lambda \in \mathbb{C}$ by

$$M[u](\lambda) = \int_0^\infty r^{-\lambda} u(r) \, \frac{dr}{r}.$$

By the change of variable $\rho = e^t$, we see that

$$M[u](\lambda) = \int_{-\infty}^\infty e^{-\lambda t} u(e^t) \, dt.$$

Hence if $\lambda = \eta + i\xi$, we see that $M[u](\lambda) = \mathfrak{F}(e^{-\eta \cdot} u(e^{\cdot}))(\xi)$. Similarly if K is a cone in $\mathbb{R}^n$ with vertex 0, if ρ denotes the distance to the origin and $\Theta = \frac{x}{\rho}$, then

$$M[u](\lambda, \Theta) = \int_0^\infty \rho^{-\lambda} u(\rho\Theta) \frac{d\rho}{\rho}.$$


Thank you